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### 15.3 *Equations Relating V-Elements Arising from Equations Relating Co-ordinates*

Equation (15.4) can also be obtained in a different way. One then proceeds on the "co-ordinate equation" of motion, that is, the equation written in terms of the co-ordinates of the V-elements. Thus, if $x = \xi m$ (where $m$ = metre) and $t = \tau s$ (where $s$ = second), the co-ordinate equation is:

$$\xi = \gamma\tau^{\alpha} \qquad \tau > 0 \tag{15.8}$$

All symbols occurring in (15.8) are real numbers, that is, elements of $\Omega$. The equation (15.8) is therefore correct. It may be recalled that in mathematical analysis the function $\tau^{\alpha}$ is defined by the identity $\tau^{\alpha} \equiv e^{\alpha \ln \tau}$. It is therefore only defined for $\tau > 0$.

$e^{\alpha * (\ln \tau)} = (e^{(\ln \tau)})^{\alpha}$

For $\tau = \tau_0 > 0$ one obtains from (15.8):

$$\xi_0 = \gamma\tau_0^{\alpha} \tag{15.9}$$

Elimination of $\gamma$ from the equations (15.8) and (15.9) leads to:

$$\frac{\xi}{\xi_0} = \left(\frac{\tau}{\tau_0}\right)^{\alpha} \tag{15.10}$$

By using the results obtained in section 13 one can immediately go over from (15.10) to (15.4), which is the notation that is consistent with the axioms of dimensional analysis set forth in this paper. Indeed, we may write:

$$\frac{x}{x_0} = \frac{\xi}{\xi_0} m^0 s^0 \tag{15.11}$$

and

$$\frac{t}{t_0} = \frac{\tau}{\tau_0} m^0 s^0$$

from which follows:

$$\left(\frac{t}{t_0}\right)^{\alpha} = \left(\frac{\tau}{\tau_0}\right)^{\alpha} m^0 s^0 \tag{15.12}$$

where $x/x_0$, $t/t_0$ and $(t/t_0)^{\alpha}$ are elements of the vector space $L^0T^0$. Such elements form a field $K$ that is isomorphic with the field $\Omega$ of the real numbers.



Combining (15.10), (15.11) and (15.12) we obtain:

$$\frac{x}{x_0} = \left(\frac{t}{t_0}\right)^{\alpha} \tag{15.4}$$

and, therefore:

$$x = x_0 \left(\frac{t}{t_0}\right)^{\alpha} \tag{15.5}$$

which is also a correct notation.

Apparently physicists are not always aware of the requirement that numbers and V-elements must be kept strictly distinct. This is how it happens that often the corresponding V-elements are substituted for the co-ordinates in a co-ordinate equation like (15.8), and, as a result, (15.6) is written. This procedure is not justified by the theory of the algebraic structure of dimensional analysis. According to this theory, it is necessary first to write the co-ordinate equation (15.10) and then to make the transition to (15.4). The reason for this is that in this example the vector space $L^0T^0$ is *the only one vector space whose elements form a field K* that is isomorphic with the field $\Omega$ of the real numbers.

## 16 Epilogue

The objective of this exposition of dimensional analysis is to show how the concepts and methods of modern algebra can be made useful to the description of certain interdependences. This is a matter of importance since physics as a science stands in the need of a conceptual clarification of these relationships. As is true of every calculus, so dimensional analysis, too, needs a mathematical foundation. Such a basis is here presented by describing its algebraic structure. The mathematical foundation has been stressed, whereas the application of dimensional analysis has only been touched in passing.[1] This has been done in order to obtain a solid basis for entering upon the task of clarifying those problems of application which are still at issue.

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[1] Further considerations on the theory and applications of dimensional analysis are presented in: W. Quade, "Zur Theorie und Anwendung des Größenkalküls der Physik", *Abhandlungen der Braunschweigischen Wissenschaftlichen Gesellschaft*, XVIII (1966), pp. 15–49. Here, applications to geometry, mechanics and electrodynamics are given. The homomorphism fundamental to applications is described, and connected questions concerning the extension of a given system of units are investigated. [*Added in 1967*].

satisfactory from a mathematical point of view. This can only be done for integral values of $\alpha$.

The example given above shows that calculating with physical quantities does not obey without restrictions the same laws as calculating with real numbers. (See also the example given on pp. 188–189 concerning the logarithm of a quotient.)

## 15.2 *V-Elements Occurring in Differential Equations and their Solutions*

Let us consider the following problem in kinematics. Assume a point is moving along a straight line, which will be called the $x$-axis. Let this moving point have the abscissa $x$ at the point of time $t$. Let it be required to describe the motion satisfying the linear and homogeneous differential equation:

$$\frac{dx}{dt} = \alpha \frac{x}{t} \tag{15.2}$$

In other words, it is required that the value of the instantaneous velocity be proportional to the quotient of abscissa and time. Let the constant $\alpha$ be some real number different from zero: $\alpha \in \Omega$.

From the viewpoint of dimensional analysis the differential equation (15.2) is obviously correct, since at both sides are V-elements whose dimension is "Length/Time", that is, if we use the symbols that are conventional in physics:

$$\frac{dx}{dt} \in LT^{-1} \qquad \frac{dx}{dt}(t) = \alpha \frac{x(t)}{t} \in LT^{-1}$$

$$\alpha \frac{x}{t} \in LT^{-1} \qquad \frac{dx}{dt}, x \in T \to LT^{-1}$$

By separation of the variables the differential equation takes the form:

$$\frac{dx}{x} = \alpha \frac{dt}{t} \tag{15.3}$$

At both sides of this equation the V-elements are elements of the vector space $L^\circ T^\circ$ – that is, the vector space of the dimensionless variables. They are, therefore, elements of the field $K$ (see p. 167 and section 13).

Integration of this equation takes place within the field $K$ and is, therefore, a legitimate procedure. By integration the following solution of (15.3) is obtained:



$$\frac{x}{x_0} = \left(\frac{t}{t_0}\right)^\alpha \tag{15.4}$$

This equation represents the solution that takes the value $x = x_0$ for $t = t_0$. There is only one solution satisfying the required initial condition.

Written in this form the result is correct in the sense of dimensional analysis, since at both sides of (15.4) are elements of the vector space $L^0 T^0$. As a basis of $L^0 T^0$ we may choose, for instance, $m^0 s^0$ where $m$ means "metre" and $s$ "second". The notation (15.4) has the property that the elements of the vector spaces $L$ and $T$ can be represented by using arbitrary bases, which means that equation (15.4) is invariant with respect to a change of the bases. The requirement of invariance implies that the units of measurement do not occur explicitly in such an equation.

It should be obvious that the following notation, which can be deduced from (15.4), is also correct in the sense of dimensional analysis:

$$x = x_0 \left(\frac{t}{t_0}\right)^\alpha \tag{15.5}$$

for at both sides we then have elements of the vector space $L$.

In physical literature one frequently comes across the following notation instead of (15.5):

$$x = ct^\alpha \quad \text{where} \quad c = \frac{x_0}{t_0^\alpha} \tag{15.6}$$

However, this notation does not completely satisfy the requirements made by dimensional analysis in this one respect that a vector space with a basis $(\text{second})^\alpha$ is only defined for integral values of $\alpha$.

Now, one needs not necessarily object to the notation (15.6) if only one has a clear vision of the meaning of an equation like (15.6). For, from (15.6) one can immediately reach (15.4), and *vice versa*. From (15.6) one comes to (15.4) by writing the following equation, which follows from (15.6):

$$x_0 = ct_0^\alpha \tag{15.7}$$

and then eliminating the constant $c$ from (15.6) and (15.7). On the other hand, the transition from (15.4) to (15.6) is problematical in this one respect that the symbol $t_0^\alpha$ occurring in $c = x_0/t_0^\alpha$ is not defined unless $\alpha$ is an integer.

Finally it should be noted that by virtue of the isomorphism between $K$ and $\Omega$ an equation in elements of $\Omega$ can immediately be deduced from (14.4). In physics such an equation is called an "equation between numerical values".

EXAMPLE: [1] Let $x_1$ be the length of a simple pendulum, $x_2$ its mass, $x_3$ its periodic time, $x_4$ the acceleration of gravity. We are to find a relation that exists between these V-elements. The number of the V-elements given is $\nu = 4$. If the CGS system is used the rank of the infinite free Abelian group $\mathbf{G}_0$ is $\rho = 3$. If the function $\Phi$ describing the relation required is supposed to belong to the class $(C)$, we have, according to (14.4):

$$\Phi(x_1, \ldots, x_4) \equiv F(y_1, \ldots, y_{4-\rho'}) = \gamma e$$

For the purpose of determining the arguments of $F$ – these are the monomials $y_\kappa$ – it is, first of all, necessary to write out the system of equations corresponding to (14.1):

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -2 \end{matrix}$$

$$\begin{aligned} \varphi(x_1) &= \omega_1 \\ \varphi(x_2) &= \quad \omega_2 \\ \varphi(x_3) &= \quad\quad \omega_3 \\ \varphi(x_4) &= \omega_1 \quad -2\omega_3 \end{aligned} \tag{14.1'}$$

Its matrix is of the rank $\rho' = 3$. The system of equations corresponding to (14.2) serves for determining the exponents $\sigma_{\lambda\kappa}$ of the monomials $y_\kappa$. Its matrix is the transpose of the matrix of the system (14.1'). Hence:

$$\begin{aligned} \sigma_1 \quad\quad\quad +\sigma_4 &= 0 \\ \sigma_2 \quad\quad\quad &= 0 \\ \sigma_3 \; -2\sigma_4 &= 0 \end{aligned} \tag{14.2'}$$

Since $\nu = 4$ and $\nu - \rho' = 1$, an integral solution of (14.2') is:

$$\sigma_1 = -1, \quad \sigma_2 = 0, \quad \sigma_3 = 2, \quad \sigma_4 = 1$$

It follows from this that $y = x_1^{-1} x_2^0 x_3^2 x_4^1$ is a monomial that is an element of $K$. The result is, therefore:

$$\Phi(x_1, \ldots, x_4) = F(x_1^{-1} x_3^2 x_4) = \gamma e$$

[1] See, for instance, Drobot's paper cited in the Introduction.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$



or:

$$x_3^2 = \gamma' \frac{x_1}{x_4} \tag{14.3'}$$

We see from this that the square of the period is independent of the mass of the pendulum, proportional to its length, and inversely proportional to the acceleration of gravity.

## 15 The Notation of Equations Relating Physical Quantities [1]

There are many examples in the literature of equations relating "physical quantities" (V-elements) written in a form that does not satisfy the requirements made by the theory of the algebraic structure of dimensional analysis. In this section it will be demonstrated by means of some typical examples how this notation can be so modified as to do full justice to the requirements of mathematical theory.

### 15.1 *A Pitfall Involved in the Extraction of Roots*

From equation (14.3') we can legitimately deduce:

$$x_3 = \beta \sqrt{\frac{x_1}{x_4}} \qquad \beta \in \Omega \tag{15.1}$$

for $\sqrt{x_1/x_4}$ is an element of the vector space of time. On the other hand, it would be inconsistent with the axioms set forth in this paper to write:

$$x_3 = \beta \frac{\sqrt{x_1}}{\sqrt{x_4}} \tag{15.1'}$$

It is illegitimate to extract the square root of the numerator $x_1$ or the denominator $x_4$, since in the infinite free Abelian group $\mathbf{G}_0$ generated by the vector spaces of length, mass and time there exists no vector space an element of which would be, for instance, $\sqrt{x_1}$, that is, the square root of a length. It should be borne in mind that the "dimensions" – that is, the elements of an infinite free Abelian group – are defined for *integral exponents only*. As far as I know, hitherto no theory has been formulated which permits defining a vector space $A^\alpha$ for arbitrary values of $\alpha$ in a way that is

[1] This section was added in 1966 at the request of Frits J. de Jong.

$\nu$ number of variables, parameters
$\rho$ number of "dimensions" in the free system



This is a system of linear and homogenous equations in the $\nu$ unknowns $\sigma_\lambda$. Its matrix $(\alpha_{\mu\lambda})$ is the transpose of the matrix of the system (14.1).

Let $\rho' \leqq \rho < \nu$ be the rank of the matrix of (14.1). The matrix of (14.2) will then be of the same rank. The system of equations (14.2) now has $\nu - \rho'$ linearly independent solutions. These solutions are $\nu$-tuples of rational numbers for the elements of the matrix $(\alpha_{\mu\lambda})$ are integers. If every $\nu$-tuple is multiplied by the common denominator, and if then the $\nu$-tuple of integers so obtained is divided by the highest common factor of the elements of the $\nu$-tuples, a "minimal" $\nu$-tuple of integers is obtained, which is a solution of (14.2). There exist $\nu - \rho'$ linearly independent solutions of this sort. This is the proof of:

THEOREM I: *If $\mathbf{G}_0$ is a free Abelian group whose generating vector spaces are $A_1, \ldots, A_\rho$, and if every V-element $x_1, \ldots, x_\nu$ (where $\nu > \rho$) is an element of a vector space belonging to $\mathbf{G}_0$, then there exist $\nu - \rho'$ independent monomials:*

$$y_\kappa = \prod_{\lambda=1}^{\nu} x_\lambda^{\sigma_{\lambda\kappa}} \qquad \kappa = 1, \ldots, \nu - \rho'$$

*which are elements of the field $K$; $\rho'$ here denotes the rank of the matrix of the system of equations* (14.1).

It follows from this Theorem that arbitrary functions $F$ can be formed from the variables $y_1, \ldots, y_{\nu-\rho'}$. This can be accomplished in the way described in the preceding section, since the variables $y_1, \ldots, y_{\nu-\rho'}$ are elements of $K$; $F(y_1, \ldots, y_{\nu-\rho'})$ itself is then also an element of $K$. If in $F(y_1, \ldots, y_{\nu-\rho'})$ the corresponding monomial in the $x_\lambda$ is substituted for every $y_\kappa$ (where $\kappa = 1, \ldots, \nu - \rho'$), a function of the V-elements $x_1, \ldots, x_\nu$ arises:

$$\Phi(x_1, \ldots, x_\nu) \equiv F(y_1, \ldots, y_{\nu-\rho'}) \tag{14.3}$$

which is an element of $K$. The function so defined on the V-set is not an arbitrary function of the arguments $x_1, \ldots, x_\nu$ but a function of a function, which arises from $F(y_1, \ldots, y_{\nu-\rho'})$ by substitution.

Let us label the functions $\Phi(x_1, \ldots, x_\nu)$ so obtained *functions of the class* $(C)$. The functions of this class are characterized by their being elements of $K$ and by the fact that they arise from an arbitrary function $F(y_1, \ldots, y_{\nu-\rho'}) \in K$ defined on $K$ by substitution of the monomials in the $x_1, \ldots, x_\nu$ belonging to the field $K$ for the arguments $y_\kappa$.

If $\Phi(x_1, \ldots, x_\nu)$ is a given function of the class $(C)$, then according to Theorem I there exist exactly $\nu - \rho'$ monomials $y_\kappa = \prod_{\lambda=1}^{\nu} x_\lambda^{\sigma_{\lambda\kappa}}$ belonging



to the field $K$ that can be formed from the $\nu$ given arguments $x_1, \ldots, x_\nu$. As $\Phi$ belongs to the class $(C)$, we have $\Phi(x_1, \ldots, x_\nu) \in K$, and a function $F(y_1, \ldots, y_{\nu-\rho'}) \in K$ exists such that (14.3) is valid. With this the $\pi$-Theorem – also called Buckingham's Theorem – has been proved:

THEOREM II ($\pi$-THEOREM): *Let $\Phi$ be a function of the class* $(C)$. *Then it is true that $\Phi(x_1, \ldots, x_\nu) \in K$, and there exist exactly $\nu - \rho'$ independent monomials $y_\kappa = \prod_{\lambda=1}^{\nu} x_\lambda^{\sigma_{\lambda\kappa}}$ belonging to the field K, such that*:

$$\Phi(x_1, \ldots, x_\nu) = F(y_1, \ldots, y_{\nu-\rho'})$$

*The $\nu$-tuples of exponents $(\sigma_{1\kappa}, \ldots, \sigma_{\nu\kappa})$ defining these monomials are the $\nu - \rho'$ linearly independent solutions of the system of equations* (14.2) *referred to in Theorem I, where $\rho'$ denotes the rank of the matrix of this system of equations.*

NOTE: With $(C)$ an extensive class of functions defined on the set of V-elements is described. Is it possible to define yet other classes of functions on this set? One may answer to this that on the ground of the axioms monomials are indeed defined on the V-set, but no polynomials (that is, sums of monomials) since the V-elements do not form a ring (see section 3). One can even show that among the set of V-elements belonging to the vector spaces of $\mathbf{G}_0$ the set $K$ is the one and only subset whose elements form a ring. That is why of all functions which are definable on the V-set only those of the class $(C)$ have been drawn into the consideration.

The functions of the class $(C)$ are mainly used in dimensional analysis. Let $\psi(x_1, \ldots, x_\nu)$ be a function which is an element of some vector space belonging to $\mathbf{G}_0$. If $\psi$ is divided by a basis of this vector space, a function $\Phi(x_1, \ldots, x_\nu)$ arises which is an element of $K$. That is why in considering functions one can confine oneself to those functions that are elements of $K$. If, furthermore, the function $\Phi$ is supposed to belong to the class $(C)$, then it follows from the $\pi$-Theorem that:

$$\Phi(x_1, \ldots, x_\nu) \equiv F(y_1, \ldots, y_{\nu-\rho'})$$

With this $\Phi$ is reduced to a function $F$ whose arguments belong to the field $K$. Since $F(y_1, \ldots, y_{\nu-\rho'})$ is itself an element of $K$, we may write:

$$\Phi(x_1, \ldots, x_\nu) \equiv F(y_1, \ldots, y_{\nu-\rho'}) = \gamma e \tag{14.4}$$

where $\gamma \in \Omega$ and $e$ is the unit element of $K$. *Every physical equation can be put in the form* (14.4), *provided that $\Phi$ belongs to the class* $(C)$.

If a function of $(x_1, x_2)$ is to be defined on a subset $T$ of $K \times K$, this can be done in the same way as for functions of one variable. By virtue of $\Phi$ the subset $T$ is mapped onto a subset $\Theta$ of $\Omega \times \Omega$. If an arbitrary function $\eta = f(\xi_1, \xi_2)$ is defined on $\Theta$, then $\Theta$ is mapped onto a subset $\Omega''$ of $\Omega$ by this function. Finally, $\Omega''$ is mapped onto a subset $K''$ of $K$ by $\varphi^{-1}$. This mapping of $T$ onto $K''$ can be illustrated by means of the following diagram:

$$\begin{tikzcd} T \arrow[r, "\Phi"] \arrow[d, "F"'] & \Theta \arrow[d, "f"] \\ K'' & \Omega'' \arrow[l, "\varphi^{-1}"'] \end{tikzcd} \qquad \begin{array}{l} T \subset K \times K,\ K'' \subset K \\ \Theta \subset \Omega \times \Omega,\ \Omega'' \subset \Omega \end{array}$$

With this every pair of elements $(x_1, x_2) \in T$ is related to one and only one element $y \in K''$. This means that a function $y = F(x_1, x_2)$ is defined on $T$. We have:

$$F = \varphi^{-1} f \Phi \tag{13.7}$$

since it is true that:

$$y = \varphi^{-1}(\eta) = \varphi^{-1} f(\xi_1, \xi_2) = \varphi^{-1} f \Phi(x_1, x_2)$$

If, conversely, a mapping $y = F(x_1, x_2)$ of $T$ onto $K''$ is given and if $\Theta$ and $\Omega''$ are the images of $T$ and $K''$ respectively, then $y = F(x_1, x_2)$ defines a mapping:

$$\eta = \varphi(y) = \varphi F(x_1, x_2) = \varphi F \Phi^{-1}(\xi_1, \xi_2)$$

of $\Theta$ onto $\Omega''$ where $\Phi^{-1}(\xi_1, \xi_2)$ is the inverse of (13.6).

A way of reasoning corresponding to that used in the case of functions of one variable leads to:

$$F(x_1, x_2) = F(\xi_1 e, \xi_2 e) = \varphi^{-1}\{f(\Phi(\xi_1 e, \xi_2 e))\} =$$
$$= \varphi^{-1}\big(f(\xi_1, \xi_2)\big) = f(\xi_1, \xi_2)e$$

As a final result we obtain:

$$F(\xi_1 e, \xi_2 e) = f(\xi_1, \xi_2)e$$

These results can immediately be transposed to functions of more than two independent variables. So we obtain for $x_1, \ldots, x_\nu \in K$:

$$F(x_1, \ldots, x_\nu) = F(\xi_1 e, \ldots, \xi_\nu e) = f(\xi_1, \ldots, \xi_\nu)e.$$

$\varphi(x_1^{\sigma_1}) + \varphi(x_2^{\sigma_2}) + \cdots + \varphi(x_\nu^{\sigma_\nu}) = \sum_{\lambda=1} \varphi(x_\lambda^{\sigma_\lambda})$
$= \sum_{\lambda=1}^{\nu} \sigma_\lambda * \varphi(x_\lambda)$



## 14 The π-Theorem (Buckingham's Theorem)

The considerations given in what follows are preparatory to the proof of the π-Theorem.

Let $A_1, A_2, \ldots, A_\rho$ be a free system of one-dimensional vector spaces. The general element of the infinite free Abelian group $\mathbf{G}_0$ belonging to this system is $\prod_{\mu=1}^{\rho}(A_\mu)^{\alpha_\mu}$ where the exponents $\alpha_\mu$ are integers. Further, let a finite number of V-elements $x_1, x_2, \ldots, x_\nu$ be given. It is assumed that every one of these V-elements is an element of a vector space belonging to $\mathbf{G}_0$, for instance such that the V-element $x_\lambda$ (where $\lambda = 1, 2, \ldots, \nu$) is an element of the vector space $\pi_{\mu=1}^{\rho}(A_\mu)^{\alpha_{\mu\lambda}}$. $\prod_{\mu=1}^{\rho} (A_\mu)^{\alpha_{\mu\lambda}} \ni x_\lambda$

*It is required to construct monomials:*

$$y = \prod_{\lambda=1}^{\nu} x_\lambda^{\sigma_\lambda} \qquad \sigma_1, \ldots, \sigma_\nu \text{ integers,}$$

$= \mathbb{Z}^\rho,\ \varphi \in G_0 \to \Gamma$

*which are elements of K.*

Monomials of this type are obtained as follows. First, the image of $x_\lambda$ in the additive Abelian group $\Gamma$ is determined. This image is given by:

$$\varphi(x_\lambda) = \sum_{\mu=1}^{\rho} \alpha_{\mu\lambda}\omega_\mu \qquad \lambda = 1, \ldots, \nu \tag{14.1}$$

where the $\omega_\mu$ have the meaning as has been indicated in section 10: they form a basis of the additive Abelian group $\Gamma$. The $\varphi(x_\lambda)$ themselves are elements of $\Gamma$. The elements of the matrix occurring in (14.1) are integers. This matrix has $\nu$ rows and $\rho$ columns. If $\nu > \rho$, then the rank of the matrix is $\rho' \leqq \rho$.

The image of the monomial $y = \prod_{\lambda=1}^{\nu} x_\lambda^{\sigma_\lambda}$ in $\Gamma$ is:

$$\varphi(y) = \sum_{\lambda=1}^{\nu} \sigma_\lambda \varphi(x_\lambda)$$

Making use of (14.1) we obtain:

$$\varphi(y) = \sum_{\lambda=1}^{\nu} \sigma_\lambda\Big(\sum_{\mu=1}^{\rho} \alpha_{\mu\lambda}\omega_\mu\Big) = \sum_{\mu=1}^{\rho}\Big(\sum_{\lambda=1}^{\nu} \alpha_{\mu\lambda}\sigma_\lambda\Big)\omega_\mu$$

It follows from this that $y$ is an element of $K$ if and only if the image of $y \in \Gamma$ is the zero element of the additive Abelian group $\Gamma$. Hence, it is true for monomials of this type that:

$$\sum_{\lambda=1}^{\nu} \alpha_{\mu\lambda}\sigma_\lambda = 0 \qquad \mu = 1, \ldots, \rho \tag{14.2}$$

$\varphi(y) = \varphi\big(\prod^{\nu} x_\lambda^{\sigma_\lambda}\big) = \varphi\big(x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_\nu^{\sigma_\nu}\big) =$

If, conversely, a mapping of $K'$ onto $K''$ defined by the function $y = F(x)$ is given and if $\Omega'$ and $\Omega''$ are the images of $K'$ and $K''$ produced by $\varphi$, then by virtue of $y = F(x)$ every element of $\Omega'$ is related to one and only one element of $\Omega''$. This means that a function $\eta = f(\xi)$ is defined on $\Omega'$. For this function it is true that:

$$f = \varphi F \varphi^{-1}$$

since the following holds good:

$$\eta = \varphi(y) = \varphi F(x) = \varphi F \varphi^{-1}(\xi)$$

According to (13.1), (13.3) and (13.4) we may write:

$$F(x) = F(\xi e) = \varphi^{-1}\{f(\varphi(\xi e))\} = \varphi^{-1}\{f(\xi)\}$$

The relation $\varphi^{-1}(\xi) = \xi e$ following from (13.3) gives the final result:

$$F(\xi e) = f(\xi) e \tag{13.5}$$

If the element $x \in K$ is identified with the number $\xi \in \Omega$ – corresponding to the isomorphism existing between $K$ and $\Omega$ – then the element $y = F(x) \in K$ is to be identified with the number $\eta = f(\xi) \in \Omega$. This follows immediately from (13.5).

The meaning of formula (13.5) may now be illustrated by two examples.

(a) If the basis $e$ is taken as "radian", then it is true that:

$$\sin(\xi e) = (\sin \xi) e$$

Strictly speaking, at the left hand side of this equation a symbol other than "sin" should have been written, since we are here concerned with the function $\varphi^{-1}\{\sin(\varphi[\xi e])\}$. It is nevertheless conventional to employ the same symbol on both sides, in the same way as the same symbol is used for $\sin z$ ($z$ complex) as for $\sin x$ ($x$ real).

(b) The second example shows that in applying (13.5) it must not be forgotten that this relation is only valid for elements of $K$. Let $a_1 = \alpha_1 a_0$ and $a_2 = \alpha_2 a_0$ be two V-elements of the vector space $A$ of the length. The quotient of these two elements is:

$$[a_1, a_2] = [\alpha_1 a_0, \alpha_2 a_0] = \frac{\alpha_1}{\alpha_2}[a_0, a_0] = \frac{\alpha_1}{\alpha_2} e$$



The logarithm of this quotient is, according to (13.5):

$$\log[a_1, a_2] = \log\left(\frac{\alpha_1}{\alpha_2} e\right) = \log\left(\frac{\alpha_1}{\alpha_2}\right) e$$

On the other hand, one must *not* write:

$$\log[a_1, a_2] = \log a_1 - \log a_2$$

for the logarithms at the right hand side are not defined, since $a_1, a_2 \in A$ whereas $A$ and $K$ are disjoint.[1]

The formula:

$$\log(\xi e) = \log \xi + \log e$$

does not make more sense, since $\log \xi \in \Omega$ but – according to (13.5) –:

$$\log e = \log(1.e) = (\log 1) e = 0 \cdot e \in K$$

and an addition between elements of $\Omega$ and elements of $K$ is not defined.

On the other hand, the multiplication theorem of the logarithmic function does have validity for elements of $K$. Indeed, we have according to (13.5):

$$\begin{aligned}\log((\xi_1 e)(\xi_2 e)) &= \log((\xi_1 \xi_2) e) = (\log(\xi_1 \xi_2)) e = \\ &= (\log \xi_1 + \log \xi_2) e = (\log \xi_1) e + (\log \xi_2) e\end{aligned}$$

The results obtained for functions of one variable will now be extended to *functions of several variables*. It will be sufficient to describe the procedure for functions of *two* variables, since this procedure can easily be extended to functions of more than two variables.

Let us now consider the set $K \times K$, that is, the set of all pairs of elements of $K$. By virtue of the mapping $\varphi$ of $K$ onto $\Omega$ defined by (13.1) the set $K \times K$ of pairs is mapped onto the set of pairs $\Omega \times \Omega$:

$$(x_1, x_2) \to (\xi_1, \xi_2)$$

13.3

Let this one-to-one mapping be represented by:

$$\Phi(x_1, x_2) = (\xi_1, \xi_2) \tag{13.6}$$

[1] The equation $\log(a_1 a_2) = \log a_1 + \log a_2$ is correct if and only if both $a_1$ and $a_2$ are elements of the field $K$. However, $a_1 a_2$ can be an element of $K$ without $a_1$ and $a_2$ being elements of $K$. In this case $\log(a_1 a_2)$ is defined: it is an element of $K$, but then $\log a_1$ and $\log a_2$ are not defined. [*Added in 1966*].

argument functions of elements of $K$ – that is, functions of "dimensionless variables" – will be considered since these functions are important for dimensional analysis. Polynominals in one variable will be examined first, for in this case the result can almost immediately be found.

Referring to section 9, p. 173, we write the unit element of $K$ as follows:

$$e = [a_0 b_0 \dots l_0, a_0 b_0 \dots l_0]$$

Every element $x$ of $K$ can then be represented as:

$$x = \xi e \tag{13.1}$$

Further, let:

$$P(x) = \sum_{\mu=0}^{\nu} \alpha_\mu x^\mu$$

be a polynomial in $x$ where the $\alpha_\mu$ are elements of $\Omega$. It follows from (13.1) that $x^\mu = \xi^\mu e$ since $e$ is idempotent, which means that $e^\mu = e$ (where $\mu$ is an integer). Hence:

$$P(x) = (\sum_{\mu=0}^{\nu} \alpha_\mu \xi^\mu) e = P(\xi) e \tag{13.2}$$

Here $P(\xi)e$ is again an element of $K$. If according to (13.1) $x$ can be identified with $\xi \in \Omega$, then according to (13.2) $P(x)$ can be identified with $P(\xi) \in \Omega$.

This result can immediately be extended to the rational functions in $x$ since any such function can be represented as a quotient of two polynomials. It remains also valid for rational functions in several variables. For instance, if $Q(x_1, x_2)$ where $x_1 = \xi_1 e$ and $x_2 = \xi_2 e$ is such a function, then the following equation, corresponding with (13.2), is true:

$$Q(x_1, x_2) = Q(\xi_1, \xi_2) e$$

Proceeding from polynomials in one independent variable it can further be shown that this result can be extended to functions which on a segment of the real variable $\xi$ can be uniformly approximated by polynomials. According to Weierstrass' Approximation Theorem this is, for instance, true for functions which are continuous on a segment. The result can be extended to analytic functions in one or more variables by virtue of the fact that these functions can be written as power series. But since these procedures are only effective for special classes of functions, we will not consider them any further.

well, modulo the type of P, Q



In the exposition which follows a procedure is described that is feasible for every function. It is based on the fact that every function can be understood to mean a "mapping", and it makes use of the isomorphism that exists between $K$ and $\Omega$. This procedure will now be described, first, for functions in *one* variable.

Let $x = \xi e$ be the general element of $K$. Since $K$ is isomorphic with $\Omega$, there exists a one-to-one mapping of $K$ onto $\Omega$, namely:

$$x = \xi e \to \xi$$ \ $\xi e \to \xi$

Let this mapping be represented as: $\varphi \in K \to \Omega$

$$\varphi(x) = \varphi(\xi e) = \xi \tag{13.3}$$

Now a function $F$ on a subset $K'$ of $K$ will be defined such that $F(x)$ is again an element of $K$.

For this purpose the image of $K'$ in $\Omega$ which is produced by the mapping $\varphi$ is determined first. The image of $K'$ will be denoted by $\Omega' \subset \Omega$. If $\eta = f(\xi)$ is some function of the variable $\xi$ which is defined on $\Omega'$, then $\eta = f(\xi)$ relates every number $\xi \in \Omega'$ to one and only one number $\eta \in \Omega'' \subset \Omega$. This means that $f$ is a mapping of the set $\Omega'$ onto the set $\Omega'' \subset \Omega$. Now let every element $\eta \in \Omega''$ be related to one and only one element $y \in K'' \subset K$ by virtue of the mapping $\varphi^{-1}(\eta) = y$ where $\varphi^{-1}$ is the inverse of the mapping $\varphi$ defined by (13.3) .These mappings can be represented by means of the following diagram:

$$\begin{tikzcd} K' \arrow[r, "\varphi"] \arrow[d, "F"] & \Omega' \arrow[d, "f"] \\ K'' & \Omega'' \arrow[l, "\varphi^{-1}"] \end{tikzcd}$$

$K', K'' \subset K$

$\Omega', \Omega'' \subset \Omega$

With this every element $x \in K'$ is related to one and only one element $y \in K''$, which means that a function $y = F(x)$ is defined on $K'$ so that $y$ itself is again an element of $K$. It is also true that:

$$F = \varphi^{-1} f \varphi \tag{13.4}$$

This follows from:

$$y = \varphi^{-1}(\eta) = \varphi^{-1} f(\xi) = \varphi^{-1} f \varphi(x)$$

$\omega_1' = \omega_1 \equiv \omega_1' = \sum_{\mu=1}^{3} \alpha_{1\mu}\omega_\mu$ with $\alpha_1 = (1,0,0)$



If the free system "Length, Force, Time" (*Force System*) is substituted for the basis "Length, Mass, Time" (*Mass System*), then this Force System, too, is a basis of the additive Abelian group $\Gamma$. The new generating elements $\omega_1', \omega_2', \omega_3'$ are obtained from the original $\omega_1, \omega_2, \omega_3$ according to:

$$\begin{aligned} \omega_1' &= \omega_1 \\ \omega_2' &= \omega_1 + \omega_2 - 2\omega_3 \\ \omega_3' &= \qquad\qquad \omega_3 \end{aligned}$$

The matrix $\mathbf{A}$ of this system of equations is unimodular, as should be clear from inspection. The triple of co-ordinates $(\alpha_1', \alpha_2', \alpha_3')$ with reference to the basis $\omega_1', \omega_2', \omega_3'$ results from:

$$\begin{aligned} \alpha_1' &= \alpha_1 - \alpha_2 \\ \alpha_2' &= \qquad \alpha_2 \\ \alpha_3' &= \qquad 2\alpha_2 + \alpha_3 \end{aligned}$$

The matrix of this system of equations is $(\mathbf{A}^{-1})^{\mathrm{T}}$, that is, the transpose of $\mathbf{A}^{-1}$. The matrix $(\mathbf{A}^{-1})^{\mathrm{T}}$ is unimodular (see section 11).

It should be noticed that, for instance, "Length, Energy, Time", or "Velocity, Impulse, Power", or "Length, Mass, Velocity" form also a basis. On the other hand, "Length, Mass, Acceleration" do not form a basis, since the corresponding transformation matrix is not unimodular: the value of its determinant is $-2$. Let us not proceed with mentioning still more possible bases. It may again be pointed out that there are infinitely many of them.

Now we are going to examine *additive Abelian subgroups* of $\Gamma$. Only additive Abelian subgroups with less than three generating vector spaces will be considered here.

Let $\Delta$ be a subset of $\Gamma$. It is demonstrated in linear algebra that the following statement is true: *If for* $\omega' \in \Delta$ and $\omega'' \in \Delta$, $\omega' - \omega'' \in \Delta$ *is also valid, then* $\Delta$ *is a subgroup of* $\Gamma$.[1] It is therefore possible immediately to ascertain that the subsets of $\Gamma$ to be discussed in the following are additive Abelian subgroups.

[1] For further guidance in submodules the reader may be referred to Chatelet, Vol. I, pp. 194 ff.



(a) The elements:

$$\omega = \alpha_1\omega_1 + \alpha_3\omega_3 \qquad (\alpha_1, \alpha_3 \text{ integers})$$

$L^\alpha T^\gamma$

form a subgroup of $\Gamma$ where $\omega_1$ and $\omega_3$ – corresponding to the vector spaces of length and time – are the generating elements. By means of the elements of this subgroup it is possible to represent not only the vector spaces that belong to geometry, but also those of velocity, acceleration, angular velocity, angular acceleration, etc. This subgroup suffices for dimensional analysis in *kinematics*.

(b) The elements:

$$\omega = \alpha_1'\omega_1' + \alpha_2'\omega_2' \qquad (\alpha_1', \alpha_2' \text{ integers})$$

where $\omega_1'$ and $\omega_2'$ have the above-mentioned meaning, form a subgroup of $\Gamma$. The two generating elements of this additive Abelian subgroup correspond to the vector spaces of length and force. By means of these two vector spaces it is possible to represent not only the vector spaces that belong to geometry, but also those of force, moment, energy, etc. This subgroup suffices for dimensional analysis in *statics*.

(c) By means of the additive Abelian subgroup whose generating elements correspond to the vector spaces of energy and time the following vector spaces among others can be represented: plane and solid angle, time, angular velocity and angular acceleration, energy, moment, power. This subgroup suffices for dimensional analysis in describing the *mechanics of rotary movements*.

At first sight it may seem amazing that, for instance, the plane angle occurs in this subgroup notwithstanding that the vector space of length, which is required for its definition, is not contained in the additive Abelian subgroup. This ostensible contradiction can be explained as follows: An additive Abelian subgroup must never be considered by itself, but it must always be taken as a *part of the whole additive Abelian group*. Now, every subgroup of $\Gamma$ contains the zero element of $\Gamma$. Hence, the additive Abelian subgroup under consideration contains also the plane angle.

## 13 Functions of Elements of the Field $K$

In physics the V-elements of the field $K$ are labelled "dimensionless variables" or "dimensionless quantities"; the expressions "dimensionless products" and "dimensionless numbers" are also used. In the following the "gram" as basis of the second vector space, the "second" as basis of the third vector space.

Applying the above developed theory to mechanics (excluding the theory of gravitation) we will indicate these one-dimensional vector spaces by $A, B, C$. This means that $A, B, C$ are the generating vector spaces of the free system. Let their basis be, respectively, $a_0, b_0, c_0$ (centimetre, gram, second). The elements of the group $G$ – that is, the non-zero generalized V-elements – are the classes represented by:

$$[\xi a_0^\alpha b_0^\beta c_0^\gamma, \xi' a_0^{\alpha'} b_0^{\beta'} c_0^{\gamma'}]$$

where $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are natural numbers and $\xi\xi' \neq 0$. For instance, $[\xi a_0^2 b_0^2 c_0, \xi' a_0 b_0 c_0^3]$ denotes a *force*, and $(\xi a_0^3 b_0^2 c_0, \xi' a_0 b_0 c_0^3]$ an *energy*.

The elements of the group $G_0$, which is homomorphic to $G$, are the classes:

$$[a_0^\alpha b_0^\beta c_0^\gamma, a_0^{\alpha'} b_0^{\beta'} c_0^{\gamma'}] \quad \alpha, \beta, \gamma, \alpha', \beta', \gamma' \text{ natural numbers}$$

Those elements of $G$ that have the unit element of $G_0$ as their image – these are the classes $[\xi a_0 b_0 c_0, \xi' a_0 b_0 c_0]$ – form an invariant subgroup $H$ of $G$. As has been demonstrated in section 7, Theorem II, the invariant subgroup $H$ can be enlarged to a field $K$ by attaching the zero element $[0 \cdot a_0 b_0 c_0, \xi' a_0 b_0 c_0]$ where $\xi' \neq 0$, to it. Since $K$ is isomorphic to $\Omega$, the elements of $K$ may be identified with numbers. More specifically, the element $[\xi a_0 b_0 c_0, \xi' a_0 b_0 c_0] \in K$ may be identified with the number $\xi/\xi' \in \Omega$. Elements of $K$ are, for instance, the plane angle and the solid angle discussed in section 8.

That physical concepts of different character can correspond to V-elements in the same vector space is demonstrated by the following example referring to the concepts of *energy* and *moment*. If the moment is defined by the equation:

Force × length = Moment

$$\text{Energy} = \text{Moment} \times \text{Angle}$$

then the representation of energy and angle by their V-elements yields:

$$[\xi a_0^3 b_0^2 c_0, \xi' a_0 b_0 c_0^3]$$

for the V-element corresponding to the moment.

Hence, the V-elements "energy" and "moment" are $R_3$-equivalent, for both have the class $[a_0^3 b_0^2 c_0, a_0 b_0 c_0^3] \in G_0$ as their image. They are, therefore, elements of one and the same vector space – a property that may seem implausible to a physicist.



Special importance is to be attached to examining the infinite free Abelian group $\mathbf{G}_0$ and the additive Abelian group $\Gamma$. Such an examination yields statements on the "systems of units" which apply to mechanics. The vector spaces $A, B, C$ are the generating vector spaces of $\mathbf{G}_0$. By the mapping (10.1) the number triples:

$$\omega_1 = (1, 0, 0), \qquad \omega_2 = (0, 1, 0), \qquad \omega_3 = (0, 0, 1)$$

are related to the generating vector spaces $A, B, C$, that form a basis of the additive Abelian group $\Gamma$. Hence, every element $\omega \in \Gamma$ can be represented by:

$$\omega = \sum_{\nu=1}^{3} \alpha_\nu \omega_\nu \qquad (\alpha_1, \alpha_2, \alpha_3 \text{ integers})$$

In the first column of Table 3.1 the triples of co-ordinates $(\alpha_1, \alpha_2, \alpha_3)$ to which some vector spaces generated by $A, B, C$ correspond are specified (*Mass System*).

TABLE 3.1

*Table of co-ordinate triples of vector spaces*

| | *Mass System* | | | *Force System* | | |
|---|---|---|---|---|---|---|
| | cm | g | s | cm | cmgs$^{-2}$ | s |
| | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_1'$ | $\alpha_2'$ | $\alpha_3'$ |
| Length | 1 | 0 | 0 | 1 | 0 | 0 |
| Area | 2 | 0 | 0 | 2 | 0 | 0 |
| Volume | 3 | 0 | 0 | 3 | 0 | 0 |
| Plane Angle | 0 | 0 | 0 | 0 | 0 | 0 |
| Solid Angle | 0 | 0 | 0 | 0 | 0 | 0 |
| Time | 0 | 0 | 1 | 0 | 0 | 1 |
| Velocity | 1 | 0 | −1 | 1 | 0 | −1 |
| Acceleration | 1 | 0 | −2 | 1 | 0 | −2 |
| Angular Velocity | 0 | 0 | −1 | 0 | 0 | −1 |
| Angular Acceleration | 0 | 0 | −2 | 0 | 0 | −2 |
| Mass | 0 | 1 | 0 | −1 | 1 | 2 |
| Force | 1 | 1 | −2 | 0 | 1 | 0 |
| Energy | 2 | 1 | −2 | 1 | 1 | 0 |
| Moment | 2 | 1 | −2 | 1 | 1 | 0 |
| Impulse | 1 | 1 | −1 | 0 | 1 | 1 |
| Power | 2 | 1 | −3 | 1 | 1 | −1 |

where the $\alpha'_\nu$, too, are integers. This can be shown in the following way: It follows from (11.1) where the $\alpha_\mu$ are integers, and from (11.3) where by virtue of the unimodularity of $\mathbf{A}^{-1}$ the $\alpha'_{\mu\nu}$ are also integers, that:

$$\omega = \sum_{\mu=1}^{\rho} \alpha_\mu \omega_\mu = \sum_{\mu=1}^{\rho} \alpha_\mu \Big( \sum_{\nu=1}^{\rho} \alpha'_{\mu\nu} \omega'_\nu \Big) =$$

$$= \sum_{\nu=1}^{\rho} \Big( \sum_{\mu=1}^{\rho} \alpha_\mu \alpha'_{\mu\nu} \Big) \omega'_\nu = \sum_{\nu=1}^{\rho} a'_\nu \omega'_\nu$$

Hence, the $\alpha'_\nu$ are also integers. By this the following Theorem has been demonstrated:

THEOREM I: *If $\omega_1, \ldots, \omega_\rho$ is a basis of the additive Abelian group $\Gamma$ then $\omega'_1, \ldots, \omega'_\rho$ is another basis if and only if the matrix $\mathbf{A} = (\alpha_{\nu\mu})$ in:*

$$\omega'_\nu = \sum_{\mu=1}^{\rho} \alpha_{\nu\mu} \omega_\mu \qquad \nu = 1, \ldots, \rho$$

*is unimodular.*

The equations (11.4) and (11.5) indicate how the integral co-ordinates of a lattice point are transformed if the basis is changed. The matrix in (11.4) is the transpose of $\mathbf{A}$, and the matrix in (11.5) is the transpose of $\mathbf{A}^{-1}$.

Let $\mathbf{A}$ be some unimodular matrix. Then by virtue of:

$$\omega' = \mathbf{A}\omega$$

every element $\omega \in \Gamma$ is related to one element $\omega' \in \Gamma$. As $\omega$ is a $\rho$-tuple of integers and $\mathbf{A}$ is unimodular, the $\rho$-tuple $\omega'$ contains integers only. If, conversely, $\omega'$ is an arbitrary element of $\Gamma$, then just one $\omega \in \Gamma$ is related to $\omega'$ by $\mathbf{A}\omega = \omega'$. This $\omega \in \Gamma$ is clearly $\omega = \mathbf{A}^{-1}\omega'$. Hence, $\mathbf{A}$ describes a one-to-one mapping of the additive Abelian group $\Gamma$ onto itself.

This mapping is even isomorphic, since for $\omega_1, \omega_2 \in \Gamma$ the following equation holds:

$$\mathbf{A}(\omega_1 + \omega_2) = \mathbf{A}\omega_1 + \mathbf{A}\omega_2 = \omega'_1 + \omega'_2$$

which means that the image of the sum equals the sum of the images. By this the following Theorem has been demonstrated:

THEOREM II: *The transformation $\omega' = \mathbf{A}\omega$ (where $\mathbf{A}$ is a unimodular matrix) describes an isomorphism of the additive Abelian group $\Gamma$ onto itself, that is, an automorphism.*



*There exist infinitely many bases.* This can be seen from the special matrix:

$$\mathbf{A} = \begin{pmatrix} 1, & \alpha_{12}, & \alpha_{13}, & \ldots, & \alpha_{1\rho} \\ 0, & 1, & \alpha_{23}, & \ldots, & \alpha_{2\rho} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \ldots, & 1 \end{pmatrix}$$

where the elements of the main diagonal have the value 1 whereas the elements at the right hand side of the main diagonal are arbitrary integers. This matrix is unimodular. With this an infinite subset of the unimodular matrices is indicated.

THEOREM III: *The unimodular matrices form a group.*

For if two unimodular matrices are multiplied, the result is another unimodular matrix. This is so since the determinant of a product of two matrices equals the product of the determinants of the factors. If the matrices are unimodular, the value of the product is again 1 or $-1$. Besides, the associative law is valid under multiplication of matrices: the neutral element – that is, the unit matrix $\mathbf{E}$ – and the inverse matrix $\mathbf{A}^{-1}$ of a unimodular matrix $\mathbf{A}$ are both unimodular, as has been pointed out above.

It should be noted that the transpose of a unimodular matrix is again unimodular. The same is true of the adjoint matrix of a unimodular matrix, and also of the symmetric matrix which arises if a unimodular matrix is multiplied by its transpose.

A detailed treatment of the theory of unimodular transformations will be found in the work of A. Chatelet already quoted in the Introduction.[1]

## 12 Application to Mechanics

In classical mechanics one conventional system in measuring quantities is the CGS system, where the letters CGS denote, respectively, "centimetre", "gram", and "second". Expressing the same idea in the language of algebra we substitute "V-elements" for "quantities". It is then said that the free system of one-dimensional vector spaces is made the basis of the considerations. The elements of these one-dimensional vector spaces are lengths in terms of centimetres, masses in terms of grams, and time intervals in terms of seconds. Here the "centimetre" serves as basis of the first vector space,

[1] Chatelet, Vol. I, pp. 194 ff.

Geometrically the elements of the additive Abelian group $\Gamma$ can be interpreted as a set of *lattice points* which is imbedded in a $\rho$-dimensional Euclidean space. One single point of the lattice is obtained by shifting the zero point of the $\rho$-dimensional space first by an integral multiple $\alpha_1$ of $\omega_1$, then by an integral multiple $\alpha_2$ of $\omega_2$, and so on. Every element of $\Gamma$ is related to just one point of the lattice, and conversely. Every vector $\omega$ leading to a point of the lattice can be represented by (10.2). On the ground of the one-to-one relation between an additive Abelian group $\Gamma$ and the lattice belonging to it the brief expression "lattice $\Gamma$" will occasionally be used in what follows.

## 11 Change of the Basis: Unimodular Matrix

The considerations of this section will be given by reference to the additive Abelian group $\Gamma$ only. By virtue of the isomorphism of $\mathbf{G}_0$ with $\Gamma$ the results obtained can immediately be transposed from $\Gamma$ to $\mathbf{G}_0$.

Let $\omega_1, \ldots, \omega_\rho$ be a basis of $\Gamma$. Every element $\omega$ of $\Gamma$ can then uniquely be represented by:

$$\omega = \sum_{\mu=1}^{\rho} \alpha_\mu \omega_\mu \tag{11.1}$$

where the $\alpha_\mu$ are integers.

Let $\omega'_1, \ldots, \omega'_\rho$ be another basis of $\Gamma$. According to (11.1) every element of the second basis can be written as a linear combination of the elements of the first basis:

$$\omega'_\nu = \sum_{\mu=1}^{\rho} \alpha_{\nu\mu} \omega_\mu \qquad \nu = 1, \ldots, \rho \tag{11.2}$$

where the $\alpha_{\nu\mu}$ are integers. The $\rho$-tuples $(\alpha_{\nu 1}, \ldots, \alpha_{\nu\rho})$ are the co-ordinates of the $\omega'_\nu$. Since the $\omega'_\nu$ are assumed to form a basis, which implies that they are linearly independent, the corresponding $\rho$-tuples are also linearly independent. Hence, the matrix $(\alpha_{\nu\mu})$ is regular. In the same way, every element $\omega_\mu$ of the first basis can be written as a linear combination of the elements of the second:

$$\omega_\mu = \sum_{\nu=1}^{\rho} \alpha'_{\mu\nu} \omega'_\nu \qquad \mu = 1, \ldots, \rho \tag{11.3}$$

where the $\alpha'_{\mu\nu}$ are also integers. The matrix $(\alpha'_{\mu\nu})$, too, is regular. Combining (11.2) and $\omega = \sum_{\nu=1}^{\rho} \alpha'_\nu \omega'_\nu$ we obtain:



$$\omega = \sum_{\nu=1}^{\rho} \alpha'_\nu \sum_{\mu=1}^{\rho} \alpha_{\nu\mu} \omega_\mu = \sum_{\mu=1}^{\rho} \Big( \sum_{\nu=1}^{\rho} \alpha_{\nu\mu} \alpha'_\nu \Big) \omega_\mu$$

On the other hand, we have (11.1): $\omega = \sum_{\mu=1}^{\rho} \alpha_\mu \omega_\mu$. As the $\omega_\mu$ are linearly independent, we obtain from the last-mentioned two representations of $\omega$:

$$\alpha_\mu = \sum_{\nu=1}^{\rho} \alpha_{\nu\mu} \alpha'_\nu \qquad \mu = 1, \ldots, \rho \tag{11.4}$$

Analogously, we get from (11.3), (11.1) and $\omega = \sum_{\nu=1}^{\rho} \alpha'_\nu \omega'_\nu$:

$$\alpha'_\nu = \sum_{\mu=1}^{\rho} \alpha'_{\mu\nu} \alpha_\mu \qquad \nu = 1, \ldots, \rho \tag{11.5}$$

Consequently the matrix $(\alpha'_{\mu\nu})$ is the inverse of the matrix $(\alpha_{\nu\mu})$. If we write $(\alpha_{\nu\mu}) = \mathbf{A}$, then $(\alpha'_{\mu\nu}) = \mathbf{A}^{-1}$.

It follows from $\mathbf{A}\mathbf{A}^{-1} = \mathbf{E}$ (where $\mathbf{E}$ is the unit matrix) that

$$|\mathbf{A}| \times |\mathbf{A}^{-1}| = |\mathbf{E}| = 1$$

where $|\mathbf{A}|$ is the determinant of the matrix $\mathbf{A}$ and $|\mathbf{E}|$ is the determinant of the unit matrix $\mathbf{E}$.

Since the elements of the two matrices $\mathbf{A}$ and $\mathbf{A}^{-1}$ are integers, their determinants have integral values, too. But then the product of the two determinants can have the value 1 only if the values of both determinants are either 1 or $-1$.

In algebra it is conventional to call a square matrix whose elements are integers and whose determinant has the value 1 or $-1$ a *unimodular* matrix. If $\mathbf{A}$ is unimodular, then $\mathbf{A}^{-1}$ is also unimodular. This demonstrates that the following statement is true: If $\omega_1, \ldots, \omega_\rho$ and $\omega'_1, \ldots, \omega'_\rho$ are two bases of the additive Abelian group $\Gamma$, then the transformation matrix $\mathbf{A} = (\alpha_{\nu\mu})$ defined by (11.2) is unimodular. The unimodularity is therefore a necessary condition for $\omega'_1, \ldots, \omega'_\rho$ being another basis.

However, this condition is also *sufficient*. In other words, if $\omega_1, \ldots, \omega_\rho$ is a basis of the module $\Gamma$ and if $\mathbf{A} = (\alpha_{\nu\mu})$ is a unimodular matrix, then the $\rho$-tuples $\omega'_1, \ldots, \omega'_\rho$ defined by (11.2) form another basis of $\Gamma$.

Indeed, by virtue of $|\mathbf{A}| \neq 0$ the $\omega'_\nu$ are linearly independent, since the $\omega_\mu$ are so. But then every element $\omega$ of $\Gamma$ can be written as a linear combination of the $\omega'_\nu$:

$$\omega = \sum_{\nu=1}^{\rho} \alpha'_\nu \omega'_\nu$$

infinite cyclic group mentioned in Section 7. The general element of $\mathbf{G}_0$ is:

$$X = A^{\alpha}B^{\beta} \dots L^{\lambda}$$

where $(\alpha, \beta, \dots, \lambda)$ can be any $\rho$-tuple of integers. The neutral element of $\mathbf{G}_0$, that is, the image of $[a_0 b_0 \dots l_0, a_0 b_0 \dots l_0]$, is $A^0 B^0 \dots L^0$.

Theorem IV of section 7 can also be copied directly:

THEOREM X: *The elements of the quotient set $G/H$ – which are the classes of the $R_3$-equivalent elements of $G$ – form a group that is isomorphic to $\mathbf{G}_0$.*

Finally, it may be remarked that it is conventional in physics to say about the element $[\xi' a_0^{\alpha'} \dots l_0^{\lambda'}, \xi'' a_0^{\alpha''} \dots l_0^{\lambda''}] \in G$ that it has the dimension $A^{\alpha} \dots L^{\lambda}$, where:

$$\alpha = \alpha' - \alpha'', \dots, \lambda = \lambda' - \lambda''$$

According to this terminology, in physics the image of a V-element that is situated in $\mathbf{G}_0$ is labelled its "dimension". It follows from this that two V-elements having the "same dimension" belong to the same vector space and can, therefore, be added. These properties will be commented upon by means of examples in section 12.

Furthermore, it follows from Theorem IX that the "physical dimensions" form an infinite free Abelian group. As far as I know this has been pointed out by R. Fleischmann for the first time.[1]

## 10 Additive Representation of an Infinite Free Abelian Group

Let us consider the infinite free Abelian group $\mathbf{G}_0$ that has been introduced in the previous section. Changing the notation a little we can write its general element as follows:

$$X = \prod_{\mu=1}^{\rho} (A_\mu)^{\alpha_\mu}$$

The elements $A_1, A_2, \dots A_\rho$ are the generating vector spaces of $\mathbf{G}_0$. The exponents $\alpha_\mu$ are integers. If the generating vector spaces are kept unchanged, each element $X \in \mathbf{G}_0$ can be related to a $\rho$-tuple of integers:

$$\omega = (\alpha_1, \alpha_2, \dots, \alpha_\rho)$$

If $\omega$ is understood to mean an image of $X$: $\varphi \in G_0 \to \mathbb{Z}^\rho = \Gamma$

$$\omega = \varphi(X) \tag{10.1}$$

[1] Fleischmann, "Die Struktur ...", already quoted in the Introduction.



and if $\Gamma$ denotes the set of the $\rho$-tuples $\omega$ of integers, then the mapping of the sets $\mathbf{G}_0$ and $\Gamma$ thus described is one-to-one. Every element $X$ in $\mathbf{G}_0$ is mapped onto one and only one $\rho$-tuple $\omega \in \Gamma$ of integers, and conversely. For instance, the element $A_1^1 A_2^0 \dots A_\rho^0$ is thus mapped onto the $\rho$-tuple $(1, 0, \dots, 0)$.

THEOREM I: *The set $\Gamma$ of the $\rho$-tuples $\omega$ of integers forms an infinite free additive Abelian group (module) which is isomorphic with the group $\mathbf{G}_0$.*

PROOF: Let the operation of multiplication in $\mathbf{G}_0$ correspond to the operation of addition in the image $\Gamma$. If $X$ and $X'$ are two elements of $\mathbf{G}_0$, then their product $XX' = X''$ is also an element of $\mathbf{G}_0$. If $\omega, \omega', \omega''$ are the images of $X$, $X'$ and $X''$, then the image of the product $XX'$ is the $\rho$-tuple:

$$\omega + \omega' = \omega''$$

From this it follows by virtue of $\omega = \varphi(X)$:

$$\varphi(X) + \varphi(X') = \varphi(X'') = \varphi(XX')$$

which means that the sum of the images equals the image of the product of the corresponding elements of $G_0$. Hence, $\mathbf{G}_0$ and $\Gamma$ are isomorphic. Q.E.D.

The $\rho$-tuple $(0, 0, \dots, 0)$ corresponds to the neutral element of $\mathbf{G}_0$. If $\omega$ is the image of $X$, then the image of $X^{-1}$ is $-\omega$. If $\omega_1, \dots, \omega_\rho$ are the images of the generating vector spaces $A_1, \dots, A_\rho$ of $\mathbf{G}_0$, that is, if $\omega_\mu = \varphi(A_\mu)$, then

$$\varphi(X) = \sum_{\mu=1}^{\rho} \alpha_\mu \varphi(A_\mu)$$

or

$$\omega = \sum_{\mu=1}^{\rho} \alpha_\mu \omega_\mu$$

is the image of $X = \prod_{\mu=1}^{\rho} (A_\mu)^{\alpha_\mu}$.

Since $\omega_\mu$ is the $\rho$-tuple all co-ordinates of which (except the $\mu$th one which has the value 1) have zero value, the system $\omega_1, \dots, \omega_\rho$ consists of $\rho$ linearly independent $\rho$-tuples.

From this follows:

THEOREM II: *If the elements $A_1, \dots, A_\rho$ form a basis of $\mathbf{G}_0$, then their images $\omega_1, \dots, \omega_\rho$ are linearly independent, and they form a basis of the additive Abelian group $\Gamma$. Each element of $\Gamma$ can be represented by:*

$$\omega = \sum_{\mu=1}^{\rho} \alpha_\mu \omega_\mu \tag{10.2}$$

If the elements of the quotient set $(S\times S)/R_2$ are regarded as generalized V-elements, Theorem III of section 5, too, remains unaltered. This Theorem remains, therefore, also valid for systems with more than one generating element:

THEOREM IV: *The non-zero (generalized) V-elements form an Abelian group $G$ with respect to the multiplicative operation $(M)$.*

The following mapping of $G$-elements is now substituted for the one described in section 6:

$$f([\xi a_0^\alpha b_0^\beta \dots l_0^\lambda, \xi' a_0^{\alpha'} b_0^{\beta'} \dots l_0^{\lambda'}]) = [a_0^\alpha b_0^\beta \dots l_0^\lambda, a_0^{\alpha'} b_0^{\beta'} \dots l_0^{\lambda'}] \qquad (9.1)$$

The mapping (9.1) divides the elements of $G$ into disjoint classes. Two elements of $G$ having the same image belong to the same class and are called $R_3$-equivalent. But two images $[a_0^\alpha \dots l_0^\lambda, a_0^{\alpha'} \dots l_0^{\lambda'}]$ and $[a_0^{\alpha_1} \dots l_0^{\lambda_1}, a_0^{\alpha'_1} \dots l_0^{\lambda'_1}]$ are equal if the corresponding pairs are $R_2$-equivalent, that is, if the $\rho$ conditions:

$$\alpha-\alpha' = \alpha_1-\alpha_1', \dots, \lambda-\lambda' = \lambda_1-\lambda_1' \qquad (R_3)$$

are satisfied. This is the *equivalence relation* which is a necessary and sufficient condition for two elements in $G$ to belong to the same class. This equivalence relation is reflexive, symmetric and transitive.

With reference to the multiplicative law which is valid in $G$, the mapping (9.1) represents a *homomorphism*. This can be shown conform to the proof of Theorem I of section 6. It leads to:

THEOREM V: *The images of the elements of the group $G$ form a group $G_0$ which is homomorphic to $G_0$; this group $G_0$ is a subgroup of $G$.*

The general element of $G_0$ is the class $[a_0^\alpha \dots l_0^\lambda, a_0^{\alpha'} \dots l_0^{\lambda'}]$ where $\alpha, \dots \lambda$ and $\alpha', \dots, \lambda'$ are natural numbers. The neutral element of $G_0$ is the class $[a_0 \dots l_0, a_0 \dots l_0]$. The inverse of the general element is $[a_0^{\alpha'} \dots l_0^{\lambda'}, a_0^\alpha \dots l_0^\lambda]$.

Theorem II of section 6 can also be copied directly:

THEOREM VI: *The elements of the quotient set $G/R_3$ – that is, the classes of $G$ – form a group which is isomorphic to $G_0$. The group $G/R_3$ is called the quotient group of $G$ with reference to $(R_3)$.*

The special classes $[\xi a_0 \dots l_0, \xi' a_0 \dots l_0]$ of $G$ have the class $[a_0 \dots l_0, a_0 \dots l_0]$ as their image, the latter being the neutral element of $G_0$. As in section 6, Theorem III, the following is also valid here:

THEOREM VII: *If $G_0$ is a group which is homomorphic with $G$, then those elements of $G$ whose image is the neutral element $G_0$ form an invariant subgroup $H$ of $G$.*



Conform to section 6 it can be shown here, too, that the equivalence relation $(R_3)$ defined by the homomorphic mapping (9.1) is equivalent with:

$$x^{-1}x' \in H \qquad (x, x' \in G)$$

Hence, the quotient set $G/R_3$ can also be written $G/H$. This set $G/H$ is a group which is isomorphic to $G_0$ and, therefore, homomorphic to $G$. It is called the *quotient group* or factor group.

That the elements in a class of $G$ form a one-dimensional vector space is proved in a way analogous to section 7.

Finally, the following Theorem is valid in the same way as Theorem II of section 7:

THEOREM VIII: *If the zero element $[0\cdot a_0 b_0 \dots l_0, a_0 b_0 \dots l_0]$ is joined to the elements of the invariant subgroup $H$ of $G$, a field $K$ is obtained which is isomorphic to the field $\Omega$.*

According to this, every element of $K$ can be "identified" with a number, that is, an element of $\Omega$. The elements of $K$ are, none the less, not numbers: they are the classes $[\xi a_0 b_0 \dots l_0, \xi' a_0 b_0 \dots l_0]$.

Instead of the mapping (7.2) of section 7 we now get the mapping:

$$f([a_0^{\alpha'} b_0^{\beta'} \dots l_0^{\lambda'}, a_0^{\alpha''} b_0^{\beta''} \dots l_0^{\lambda''}]) = A^\alpha B^\beta \dots L^\lambda \qquad (9.2)$$

where $\alpha = \alpha'-\alpha''$, $\beta = \beta'-\beta''$, ..., $\lambda = \lambda'-\lambda''$.

By (9.2) the elements of $G_0$ are mapped onto the set:

$$\{A^\alpha B^\beta \dots L^\lambda\}$$

with every exponent $\alpha, \beta, \dots, \lambda$ individually running through *all* integers (not just the positive integers). If the set of the images $\{A^\alpha B^\beta \dots L^\lambda\}$ is written $\mathbf{G}_0$, we obtain the following Theorem, which corresponds to Theorem III of section 7:

THEOREM IX: *The elements of $\mathbf{G}_0$ form an infinite free Abelian group whose generating vector spaces are $A, B, \dots L$. This Abelian group $\mathbf{G}_0$ is isomorphic to $G_0$.*

The isomorphism of these two groups is demonstrated in a way corresponding to Theorem III of section 7. Since the elements of $\mathbf{G}_0$ are vector spaces that are pairwise disjoint, $\mathbf{G}_0$ is an infinite free Abelian group with the generating vector spaces $A, B, \dots, L$.

The infinite free Abelian group here represents a generalization of the

Hence, the vector space $K$ *cannot* be a generating vector space of a free system of rank 1, in contrast to the vector space $A$.

Finally, it should be recalled (see section 3) that in the relevant physical literature the basic elements of the generating vector spaces of a free system are named "basic units". According to this terminology neither the unit of the plane angle nor the unit of the solid angle can be basic units: they are "derived units".

## 9 Free Systems with Two or More Generating Vector Spaces

The contents of this section fit in with the final part of section 3, and the results that have been obtained for systems with *one* generating element in sections 4–7 will now be generalized so that they apply to systems with *more than one* generating element. The following argument being analogous to that of the previous sections, it will be kept somewhat more succinct.

If $A, B, \ldots, L$ is a *free system* of rank $\rho$, then the set:

$$N = \{A^\alpha B^\beta \ldots L^\lambda\} \qquad (\alpha, \beta, \ldots, \lambda \text{ natural numbers})$$

consists of vector spaces which are pairwise disjoint. The general element of the vector space $A^\alpha B^\beta \ldots L^\lambda$ is $\xi a_0^\alpha b_0^\beta \ldots l_0^\lambda$ with $\xi \in \Omega$. As has been pointed out in section 3, the union set containing all elements of the vector spaces of $N$ is a *commutative semi-group*.

The general element of the set $S$ that was introduced in section 4 is:

$$x = \xi a_0^\alpha b_0^\beta \ldots l_0^\lambda \quad \text{where} \quad \xi \neq 0$$

since, in forming $S$, in each vector space belonging to $N$ the zero element is left out. As in section 4, here, too, the following theorem is valid:

THEOREM I: *The elements of $S$ are regular and form, therefore, a commutative regular semi-group.*

The proof runs analogously to that of Theorem I of section 4. If we have $u, x, x' \in S$ and the equation $ux = ux'$, then it is to be proved that the equality:

$$x = x'$$

follows from the last-mentioned equation. For if the elements of $S$ on both sides of the equation $ux = ux'$ are to be equal, they must be elements of the same vector space. From this it follows, first, that $x \equiv x' \pmod{R_1}$. Moreover, $x$ and $x'$ must have the same co-ordinate with reference to some



basis of the vector space to which both $x$ and $x'$ belong, since $u$ cannot be a zero element. Hence, the equality $x = x'$ must be true. The elements of $S$ must therefore be regular and the conclusion is that $S$ is a commutative regular semi-group.

With reference to *division* it must be said that in $S$ it is only possible under restrictions. Division, that is, the solution of the equation:

$$ux = u'$$

by an element $x \in S$ is only possible if for $u \in A^\alpha B^\beta \ldots L^\lambda$ and $u' \in A^{\alpha'} B^{\beta'} \ldots L^{\lambda'}$ the $\rho$ conditions:

$$\alpha < \alpha', \beta < \beta', \ldots, \lambda < \lambda'$$

are satisfied. However, as soon as even one of these conditions is not satisfied, the division in $S$ is infeasible. This conclusion is reached in a way analogous to section 4.

Just as in the case of free systems with one generating element, so the question is now how to imbed the regular semi-group $S$ in a group. It is necessary to do so as in a group division is feasible without restrictions: there exists an inverse of every element in a group. The procedure of imbedding a regular semi-group $S$ in a group has already been described in section 5. The results formulated there remain unaltered irrespective of the rank of the free system. Hence:

THEOREM II: *The classes of pairs of V-elements – that is, the elements of the quotient set $(S \times S)/R_2$ – form an Abelian group $G$ with respect to the multiplicative operation $(M)$.*

If the general element of $G$ is written:

$$x = [\xi a_0^\alpha b_0^\beta \ldots l_0^\lambda, \xi' a_0^{\alpha'} b_0^{\beta'} \ldots l_0^{\lambda'}]$$

then the unit element of $G$ is the class:

$$[a_0 b_0 \ldots l_0, a_0 b_0 \ldots l_0]$$

and the inverse element is the class:

$$[\xi' a_0^{\alpha'} b_0^{\beta'} \ldots l_0^{\lambda'}, \xi a_0^\alpha b_0^\beta \ldots l_0^\lambda]$$

Theorem II of section 5 can be copied directly:

THEOREM III: *The regular semi-group $S$ is isomorphic to a proper subset of $G$.*

Besides these vector spaces, the concepts of "plane angle" and "solid angle" are of importance in this connection.

If a plane angle is measured by means of the arc length, its measure is the quotient of two lengths, namely, the length of the arc that is cut out of a circle by the angle under consideration where the vertex of the angle is the centre of the circle, divided by the radius of the circle.

Let $\alpha a_0$ be the length (in metres) of the arc which is cut out and $\beta a_0$ the length (in metres) of the radius of the circle. The "plane angle" is then represented by the V-element:

$$[\alpha a_0, \beta a_0] \in K$$

As it has been explained above, the V-elements representing "plane angles" form a one-dimensional vector space. The class $[a_0, a_0]$ can be taken as a basis of this vector space. When considering plane angles the basis $[a_0, a_0]$ is named 1 rad (radian) by physicists.

Two angles $[\alpha a_0, \beta a_0]$ and $[\alpha' a_0, \beta' a_0]$ are equal if and only if:

$$(\alpha a_0, \beta a_0) \equiv (\alpha' a_0, \beta' a_0) \pmod{R_2}$$

that is, if:

$$\alpha\beta' = \alpha'\beta$$

A "solid angle" can be represented as the quotient of two areas, namely the area of the surface that is cut out of a sphere by the space angle under consideration where the vertex of the angle is the centre of the sphere, divided by the square of the radius of the sphere.

Let $\alpha a_0^2$ be the area (in square metres) of the surface which is cut out and $\beta\alpha_0$ the length (in metres) of the radius of the sphere. The "solid angle" is then represented by the V-element:

$$[\alpha a_0^2, \beta^2 a_0^2] \in K$$

Now, the classes $[\alpha a_0, \beta a_0]$ and $[\alpha a_0^2, \beta^2 a_0^2]$ are $R_3$-equivalent. They belong, therefore, to one and the same vector space $K$. We have already chosen $[a_0, a_0]$ as the basis of $K$.

When considering solid angles the basis $[a_0, a_0]$ is named 1 sr (steradian) by physicists. This shows that the different denominations stem from a kind of looking at physical qualities which are ignored in dimensional analysis. Dimensional analysis takes no cognizance of any difference between "radian" and "steradian". All V-elements which are quotients of two lengths, or two



areas, or two volumes, are elements of the vector space $K$ which has $[a_0, a_0]$ as a basis.

In this connection it is to be noticed that the V-element "plane angle" can also be defined as the quotient of two areas. It is then twice the area of the sector that is cut out of a circle by the angle under consideration where the vertex of the angle is the centre of the circle, divided by the area of the square whose side-length is equal to the radius of the circle. This leads to the class $[\alpha\beta a_0^2, \beta^2 a_0^2]$, which is the same class as $[\alpha a_0, \beta a_0]$. This shows that the denomination of radian and steradian has a certain arbitrariness. This is also demonstrated by the fact that certain V-elements of $K$, as for instance, the "linear strain" and the "sine of an angle", have not been given special names, in contrast to radian and steradian.

It has been shown in section 7, Theorem II, that the field $K$ is isomorphic to the field $\Omega$, and that the elements of $K$ may therefore be identified with the corresponding elements of $\Omega$. So any plane or solid angle may be identified with a number. The same is generally true for any quotient of two lengths, two areas, or two volumes. Since V-elements in the same vector space can be added to each other, a V-element belonging to a plane angle can be added to a V-element belonging to a solid angle. In section 13 more will be said about functions of elements in the field $K$.

In physics the elements of $K$ are labelled "dimensionless quantities" or "quantities of the dimension 1". The first name expresses the idea that these V-elements are of the dimension $A^0$, the second that any such V-element can be identified with a number. In effect, the number $\alpha/\beta$ may be substituted for the class $[\alpha a_0, \beta a_0]$ without the result of the calculation being affected since, as has been shown above, the field $K$ is isomorphic to the field $\Omega$.

Occasionally one comes across the opinion that a procedure leading from elements of the field $K$ to a group would be possible. This procedure would allegedly consist of two steps. Since the elements of $K$ form a vector space as the elements of $A$ also do – see section 7, Theorem I – the first step would be to get from the elements of $K$ to a semi-group corresponding to the semi-group $S$ that has been discussed in section 4. The second step would then be to obtain a group by means of symmetricizing the semi-group. However, this is not possible since the product of two elements of $K$ yields another element of $K$, which obviously belongs to the same vector space. On the other hand, the product of two elements of $A$ belongs to a vector space different from $A$: the latter has been written $A^2$ in the argument given above. Every product of a finite number of elements of $K$ is again an element of $K$.

$[a_0, a_0][a_0, a_0] = [a_0^2, a_0^2] = [a_0, a_0]$



of $H$, we obtain:

$$[\alpha\alpha' a_0^{\mu+1}, \beta\beta' a_0^{\nu+1}] \in G$$

This element is $R_3$-equivalent to $[\alpha a_0^{\mu}, \beta a_0^{\nu}]$. It belongs, therefore, to the same vector space as $(\alpha a_0^{\mu}, \beta a_0^{\nu}]$. The same result appears if $[\alpha a_0^{\mu}, \beta a_0^{\nu}]$ is multiplied by the number $\alpha'/\beta'$, namely:

$$[\alpha\alpha' a_0^{\mu}, \beta\beta' a_0^{\nu}] = [\alpha\alpha' a_0^{\mu+1}, \beta\beta' a_0^{\nu+1}]$$

This means that, although the concepts of "number" and "element of $K$" – the latter being a class of pairs of V-elements – are defined in a different way, in dimensional analysis they may be substituted, the one for the other, without the result being affected.

The result not being affected by this substitution it has been customary, in theory as well as in practice, to "identify" the elements of $K$ with numbers.[1] By virtue of the isomorphism mentioned above this may indeed be done without fear of inconsistencies.

It is nevertheless misleading to state that the elements of $K$ *are* elements of $\Omega$. The elements of $K$ are the classes $[\alpha a_0, \beta a_0]$, that is, classes of pairs of V-elements. In addition, it is conventional to identify the unit element of $G$, that is, $[a_0, a_0]$, with the number 1. It should be noticed that this element is *idempotent*, which means that every power of this element equals the element itself.

Let us now consider the mapping:

$$f([a_0^{\mu}, a_0^{\nu}]) = A^{\lambda} \qquad \lambda = \mu - \nu \tag{7.2}$$

This is a mapping of the elements of $G_0$ onto the set of the $A^{\lambda}$ (where $\lambda = 0, \pm 1, \pm 2, \ldots$).

If we write $\mathbf{G}_0$ for the set of the images $A^{\lambda}$, the following theorem is valid:

THEOREM III: *The elements of* $\mathbf{G}_0$ *form an infinite cyclic group whose generating element is* $A$; *the group* $\mathbf{G}_0$ *is isomorphic with* $G_0$.

In effect, the mapping (7.2) is one-to-one, for two elements $[a_0^{\mu}, a_0^{\nu}]$ and $[a_0^{\mu'}, a_0^{\nu'}]$ of $\mathrm{G}_0$ are equal if $\mu - \nu = \mu' - \nu'$. Further, the image of the product of two elements in $G_0$ equals the product of the images of the factors:

$$f([a_0^{\mu}, a_0^{\nu}][a_0^{\mu'}, a_0^{\nu'}]) = f([a_0^{\mu+\mu'}, a_0^{\nu+\nu'}]) = A^{\lambda}A^{\lambda'}$$

where $\lambda = \mu - \nu$, $\lambda' = \mu' - \nu'$. It is evident that the elements $A^{\lambda}$ (where $\lambda$ is an integer) form an infinite cyclic group.

[1] See on this: Van der Waerden, pp. 54 ff., and Bourbaki, chapter II, pp. 107–108.

$G_0 = \{A^{\lambda} \mid \lambda \in \mathbb{Z}\}$



It should be noticed that the generating element $A$ is the image of $[a_0^2, a_0] \in G_0$, and that the neutral element $A^0$ is the image of $[a_0, a_0] \in G_0$.

From this and Theorem II of section 6 it follows immediately that:

THEOREM IV: *The elements of the quotient set $G/H$ – which are the classes of $G$ – form a group that is isomorphic to* $\mathbf{G}_0$.

As has been shown above, the elements of a class of $G$ form a vector space. Hence, it follows from Theorem IV that two V-elements have the same image in $\mathbf{G}_0$ if and only if they belong to the same vector space.

From the results given above it follows that a group which is isomorphic to $G$ can be obtained if the concept of *product of two groups* is applied. One of the factors is then the multiplicative group of the non-zero elements of $\Omega$, the other is the infinite cyclic group having $a_0^{\lambda}$ (where $\lambda = 0, \pm 1, \ldots$) as its elements.[1]

In physics it is conventionally said that the element $[\alpha a_0^{\mu}, \beta a_0^{\nu}] \in G$ has the "dimension $A^{\lambda}$" (where $\lambda = \mu - \nu$). In other words, the image of the element under consideration which is situated in $\mathbf{G}_0$ is labelled the "dimension" of this V-element. In this sense the dimensions form an infinite cyclic group having $A$ as its generating element. Two V-elements have, therefore, the same "dimension" if and only if they belong to the same vector space. As the elements of a vector space form an additive Abelian group, V-elements having the same dimension can be added to each other. With this a peculiarity of the dimensional analysis has been described which deserves special attention. We will go more into detail of this in the final part of the next section.

## 8 Application to Geometry

Lengths, areas and volumes are concepts belonging to geometry. For the representation of lengths, areas, and volumes V-elements are used. Let the vector space $A$ of the lengths be the generating vector space of a free system of rank 1. The vector spaces $A, A^2, A^3, \ldots$ are then disjoint. Let us choose $a_0 = 1$ metre as the basis of $A$. We can then take $a_0^2 = 1$ square metre as the basis of $A^2$, that is, the vector space of the areas. In the same way we can consider $a_0^3 = 1$ cubic metre as the basis of $A^3$, the vector space of the volumes. The vector spaces $A^4, A^5, \ldots$ will be left out of consideration here.

[1] *Cf.* Fleischmann, "Einheiteninvariante Größengleichungen, ...", p. 449. See also the Note in the first part of section 6.

let $0 = [\alpha' a_0^{\mu'}, \beta' a_0^{\nu'}]$. Now $0 \oplus [\alpha a_0^{\mu}, \beta a_0^{\nu}] = [\alpha a_0^{\mu}, \beta a_0^{\nu}]$



## 7 Vector Spaces of Generalized V-Elements

In section 1 it was required that the V-elements shall be elements of vector spaces. Now the question arises whether a similar structure can be ascribed to the generalized V-elements, that is, the elements of $G$.

THEOREM I: *The elements of every class of $G$ which is defined by the equivalence relation $(R_3)$ form a vector space under the operational rules $(V_1)$ and $(V_2)$ as stated below.*

PROOF. The elements of a class of $G$ as defined by $(R_3)$ are those elements of $G$ which have the same element of $G_0$ as their image. Now it is to be proved first that these elements can be added so that the sum of two elements in the same class is another element in the class under consideration (closure). So let us write:

$$[\alpha a_0^{\mu}, \beta a_0^{\nu}] \equiv [\alpha' a_0^{\mu'}, \beta' a_0^{\nu'}] \pmod{R_3}$$

and, therefore:

$$\mu - \nu = \mu' - \nu'$$

The sum of the two equivalent elements is defined as follows:

$$[\alpha a_0^{\mu}, \beta a_0^{\nu}] \oplus [\alpha' a_0^{\mu'}, \beta' a_0^{\nu'}] = [\alpha\beta' a_0^{\mu+\nu'} + \alpha'\beta a_0^{\mu'+\nu}, \beta\beta' a_0^{\nu+\nu'}] \qquad (V_1)$$

The sum is $R_3$-equivalent to each of the summands, that is, it belongs to the same class as the summands.

If $[\alpha a_0^{\mu}, \beta a_0^{\nu}]$ is a representative of a class of $R_3$-equivalent elements and if, moreover, $[0 \cdot a_0^{\mu}, \beta_0^{\nu}]$ with $\beta \neq 0$ is introduced into each of such classes as zero element, then it can be shown that the elements of each class of $G$ satisfy the requirements (1.1)–(1.4). Hence, the following statement is true:

$\beta a_0^{\nu}$ ?

*The elements of one and the same class of $G$ form an additive Abelian group under the addition $(V_1)$.*

Besides, let us define a scalar multiplication $(V_2)$ of an element $\rho$ of $\Omega$ with an element of $G$:

$$\rho[\alpha a_0^{\mu}, \beta a_0^{\nu}] = [\rho\alpha a_0^{\mu}, \beta a_0^{\nu}] \qquad (V_2)$$

This operation is such that it relates the element $[\alpha a_0^{\mu}, \beta a_0^{\nu}]$ to an $R_3$-equivalent element, that is, to an element of the same class (external composition). This operation clearly satisfies the requirements (2.1)–(2.4).

This shows that the elements of a class of $G$ form a "vector space". This vector space is one-dimensional. Indeed, if $[\alpha a_0^{\mu}, \beta a_0^{\nu}]$ is an arbitrary

$\mu - \nu = \mu' - \nu' \Rightarrow \mu + \nu' - \nu - \nu' = \mu - \nu$

$[x, y]$



element of the vector space and if, for instance, $[a_0^{\mu}, a_0^{\nu}]$ is chosen as basis of this vector space, then we may write according to $(V_2)$:

$[x, y] = [x', y'] \equiv xy' = x'y$

$$[\alpha a_0^{\mu}, \beta a_0^{\nu}] = \left[\frac{\alpha}{\beta} a_0^{\mu}, a_0^{\nu}\right] = \frac{\alpha}{\beta}[a_0^{\mu}, a_0^{\nu}]$$

A special place is held by those elements of $G$ which are elements of the invariant subgroup $H$, that is, the elements of the classes $[\alpha a_0, \beta a_0]$. In physics these elements are called "dimensionless variables" or "dimensionless quantities"; "dimensionless products" is also used.

THEOREM II: *If the zero element $[0 \cdot a_0, a_0]$ is joined to the elements of the invariant subgroup $H$ of $G$, a field $K$ is obtained which is isomorphic to the field $\Omega$.*

PROOF: We shall have to demonstrate first that the elements of $K$ form a commutative *ring*, which means that they have the following properties:

(a) $K$ is an Abelian group under addition; 1.1, 1.2, 1.3, 1.4

(b) multiplication in $K$ is associative and commutative;

(c) multiplication is distributive with respect to addition.

The properties (a) and (b) have already been demonstrated above. The proof of (c) follows by calculation after substitution of $[\alpha a_0, \beta a_0]$, $[\alpha' a_0, \beta' a_0], \ldots$.

The elements of $H$ are the non-zero elements of $K$. They form an Abelian group under multiplication since the invariant subgroup $H$ is a subgroup of $G$. Hence, $K$ is a commutative field.

It remains to be shown that $K$ is isomorphic to the field $\Omega$. Indeed, if the element $[\alpha a_0, \beta a_0]$ of $K$ is related to the number $\alpha/\beta$ by the mapping:

$$f([\alpha a_0, \beta a_0]) = \frac{\alpha}{\beta} \qquad (7.1)$$

where $\beta \neq 0$ by the definition of $K$, and if $a_0$ is kept fixed, then this mapping is one-to-one, since $[\alpha a_0, \beta a_0] = \alpha/\beta\, [a_0, a_0]$. From $(V_1)$ and $(V_2)$ we obtain:

$$f([\alpha a_0, \beta a_0]) + f([\alpha' a_0, \beta' a_0]) = f([(\alpha\beta' + \alpha'\beta)a_0, \beta\beta' a_0])$$

$$f([\alpha a_0, \beta a_0]) \cdot f([\alpha' a_0, \beta' a_0]) = f([\alpha\alpha' a_0, \beta\beta' a_0])$$

Hence, the mapping defined above is isomorphic. Q.E.D.

If the element $[\alpha a_0^{\mu}, \beta a_0^{\nu}]$ of $G$ is multiplied by the element $[\alpha' a_0, \beta' a_0]$

homom. preserves product



the elements of the vector spaces $A^{\nu}$. By means of (6.3), the elements of $G$ are partitioned into disjoint classes, for every element of $G$ has just one image, and two elements of $G$ are contained in the same class if they have the same image. Two elements of $G$ having the same image and, therefore, belonging to the same class will be called $R_3$-equivalent. But two images $[a_0^{\mu}, a_0^{\nu}]$ and $[a_0^{\mu'}, a_0^{\nu'}]$ are equal if and only if the pairs $(a_0^{\mu}, a_0^{\nu})$ and $(a_0^{\mu'}, a_0^{\nu'})$ are $R_2$-equivalent, that is, if $a_0^{\mu+\nu'} = a_0^{\mu'+\nu}$, or $\mu+\nu' = \mu'+\nu$, or

$$\mu - \nu = \mu' - \nu' \qquad (R_3)$$

The relation $(R_3)$ is the *equivalence relation* which is necessary and sufficient for two elements of $G$ to have the same image and, therefore, to belong to the same class. The equivalence relation $(R_3)$ is reflexive, symmetric and transitive.

If two elements of $G$, say, $[\alpha a_0^{\mu}, \beta a_0^{\nu}]$ and $[\alpha' a_0^{\mu'}, \beta' a_0^{\nu'}]$, belong to the same class, we write:

$$[\alpha a_0^{\mu}, \beta a_0^{\nu}] \equiv [\alpha' a_0^{\mu'}, \beta' a_0^{\nu'}] \ (\mathrm{mod}\ R_3)$$

Def

With reference to the law of multiplication $(M)$ which is valid for the elements of $G$, the mapping (6.3) represents a *homomorphism*, which means that the image of the product of two elements in $G$ equals the product of the images of the factors. Indeed, if $[\alpha a_0^{\mu}, \beta a_0^{\nu}]$ and $[\alpha' a_0^{\mu'}, \beta' a_0^{\nu'}]$ are two elements of $G$, then their product is:

$$[\alpha\alpha' a_0^{\mu+\mu'}, \beta\beta' a_0^{\nu+\nu'}]$$

and the image of the product is $[a_0^{\mu+\mu'}, a_0^{\nu+\nu'}] = [a_0^{\mu}, a_0^{\nu}][a_0^{\mu'}, a_0^{\nu'}]$.

Since in this homomorphism the images are again elements of $G$, we have here to do with a case of "endomorphism".

Let the set of images – that is, the set of the classes $[a_0^{\mu}, a_0^{\nu}]$ where $\mu$ and $\nu$ are natural numbers – be written $G_0$. It is a subset of $G$ and it has the following properties:

(a) the elements of $G_0$ form a semi-group which is homomorphic to $G$;

(b) the image of the neutral element of $G$ – that is, the class $[a_0, a_0]$ – is the neutral element of $G_0$;

(c) inverse elements of $G$ have inverse elements of $G_0$ as their images.

Hence, we can formulate the following:

THEOREM I: *The images of the elements of the group $G$ form a group $G_0$ which is a subgroup of $G$ and which is homomorphic to $G$.*



It can be demonstrated by calculation that the multiplicative operation $(M)$, which is valid for elements in $G$, is also valid for the classes of $G$. Consequently the set $G/R_3$ – that is, the set of the classes into which the elements of $G$ are partitioned by the equivalence relation $R_3$ – is isomorphic to the group $G_0$. So we may now state:

THEOREM II: *The elements of the quotient set $G/R_3$ – that is, the classes of $G$ – form a group which is isomorphic to $G_0$.*

The group $G/R_3$ is called the *quotient group* of $G$ with reference to $(R_3)$. Those elements in $G$ whose image is the neutral element of $G_0$ are of special importance. They are the classes $[\alpha a_0, \beta a_0]$; their set forms an *invariant subgroup* $H$ of $G$, since the following theorem is valid:

THEOREM III: *If $G_0$ is a group which is homomorphic to $G$, then those elements of $G$ whose image is the neutral element of $G_0$ form an invariant subgroup $H$ of $G$.*

PROOF: The elements of $H$ form a subgroup of $G$. For this to be true it is a necessary and sufficient condition that, with any one pair of elements $x$ and $x'$ in $H$, the product $x'x^{-1}$ be also an element of $H$. It follows indeed from $x = [\alpha a_0, \beta a_0]$ and $x' = [\alpha' a_0, \beta' a_0]$ that

$$x'x^{-1} = [\alpha' a_0, \beta' a_0][\beta a_0, \alpha a_0] = [\alpha'\beta a_0^2, \alpha\beta' a_0^2] \in H$$

Now, a subgroup of $G$ is called an invariant subgroup if it follows from $x^{-1}x' \in H$ with $x, x' \in G$ that $x'x^{-1} \in H$, and conversely. Since $G$ is an Abelian group, each subgroup of $G$ is an invariant subgroup. Q.E.D. inv.

comm.

THEOREM IV: *The equivalence relation defined by the homomorphic mapping (6.3) is equivalent to the relation:*

$$x^{-1}x' \in H \qquad (x, x' \in G)$$

For, if $x = [\alpha a_0^{\mu}, \beta a_0^{\nu}]$ and $x' = [\alpha' a_0^{\mu'}, \beta' a_0^{\nu'}]$, then it is true that:

$$x^{-1}x' = [\alpha'\beta a_0^{\mu'+\nu}, \alpha\beta' a_0^{\mu+\nu'}]$$

It is a necessary and sufficient condition for $x^{-1}x' \in H$ to be true that $\mu'+\nu = \mu+\nu'$, or:

$$\mu - \nu = \mu' - \nu'$$

But this is the equivalence relation $(R_3)$. The quotient set $G/R_3$ may, therefore, also be written $G/H$. It is a group which is isomorphic to $G_0$ and, therefore, homomorphic to $G$. This group $G/H$ is labelled *quotient group* or *factor group*.

it were true that $[xy, y] = [x'z, z]$. By reason of the regularity of the elements of $S$ it would then follow that $x = x'$, which, however, is inconsistent with the assumption $x' \neq x$. The images of the elements in $S$ form a proper regular semisubset of $G$ for the very reason that in the semi-group $S$ there exists no neutral element and, therefore, an inverse of $x$ can also not exist.

Let $x$ and $x'$ be two elements of $S$, and let their images be the classes $[xy, y]$ and $[x'z, z]$ with $y \in S$ and $z \in S$. The image of the product $xx'$ is the class $[xx'u, u]$ with $u \in S$ while the product of the images yields the class $[xx'yz, yz]$. By virtue of $(xx'u, u) \equiv (xx'yz, yz) \pmod{R_2}$, however, this is the same class. This means that the image of the product equals the product of the images. Thus we obtain:

THEOREM II: *The regular semi-group $S$ is isomorphic with a proper subset of $G$.*

Therewith the regular semi-group $S$ is imbedded in the group $G$. The elements of the quotient set $(S \times S)/R_2$ are classes of pairs of V-elements. They can be considered to be "generalized V-elements". Besides, every element of $S$ – that is, every V-element in the original sense – can be "identified" with the element of $G' \subset G$ that has been related to it, that is, with a "generalized" V-element. The word "identified" expresses the idea that for purposes of calculation these two concepts whose definitions are different can, none the less, be substituted one for the other without the result being affected. It will not, therefore, present any difficulty to denote generalized V-elements by the same symbols as V-elements in the original sense, for instance, by $x, y, \ldots$. This possibility will fairly often be used in what follows.

So the following fundamental property of the algebraic structure of dimensional analysis follows from the foregoing theorem:

THEOREM III: *The non-zero (generalized) V-elements which are mutually related by the rule of multiplication (M) form an Abelian group G.*

## 6 On the Group G

For the sake of simplicity we will examine the group $G$, to begin with, for the case of a *free system of rank* 1. If $A$ is the only generating vector space and if $a_0$ is *any one* basis of $A$, then $a_0^\nu$ is a basis of the vector space $A^\nu$.

For every element $a^{(\nu)} \in A^\nu$ we have:

$$a^{(\nu)} \equiv a_0^\nu \pmod{R_1}$$



If this equivalence relation is regarded as a *mapping* of the elements of $A^\nu$ onto the element $a_0^\nu$, then this mapping is such that each element of $A^\nu$ is related to just one element $a_0^\nu$:

$$f(a^{(\nu)}) = a_0^\nu \tag{6.1}$$

The mapping (6.1) is unique only in the direction from the original $a^{(\nu)}$ to the image $a_0^\nu$. The converse, however, is not true, since infinitely many originals belong to the image $a_0^\nu$: namely, the set of elements of the vector space $A^\nu$. This mapping is *homomorphic*, which means:

$$f(a^{(\mu)}a^{(\nu)}) = f(a^{(\mu)})f(a^{(\nu)})$$

that is to say, the image of the product equals the product of the images. Indeed, the image of $a^{(\mu)}a^{(\nu)}$ is $a_0^{\mu+\nu}$. The statement then follows from $f(a^{(\mu)}) = a_0^\mu$ and $f(a^{(\nu)}) = a_0^\nu$.

NOTE: If in the set of vector spaces $A, A^2, \ldots$ a multiplication is defined by the statement:

$$A^\mu A^\nu = A^{\mu+\nu} \qquad \mu, \nu \text{ natural numbers}$$

then an *isomorphism* exists between the elements of the set $a_0, a_0^2, \ldots$ and those of the set $A, A^2, \ldots$. If $A^\nu$ is the image of $a_0^\nu$:

$$f(a_0^{(\nu)}) = A^\nu \qquad \nu \text{ natural number} \tag{6.2}$$

then this mapping is reversibly unique if the basis $a_0$ remains unchanged, and the product of the images equals the image of the products:

$$f(a_0^\mu)f(a_0^\nu) = f(a_0^{\mu+\nu})$$

(All mappings are denoted by $f$, which does not, of course, mean that these mappings are all the same.) In examining the structure it does not make any difference whether the set of the bases $a_0, a_0^2, \ldots$ or the set of the vector spaces $A, A^2, \ldots$ is considered.

The question now is to extend the mapping (6.1), which has been described for the elements of the vector $A^\nu$, to the elements of $G$. Let $[\alpha a_0^\mu, \beta a_0^\nu]$ be an element of $G$, that is, a class of $R_2$-equivalent pairs of V-elements. This element of $G$ is required to have the class $[a_0^\mu, a_0^\nu]$ as its image. So let us define:

$$f([\alpha a_0^\mu, \beta a_0^\nu]) = [a_0^\mu, a_0^\nu] \tag{6.3}$$

This mapping corresponds to the mapping (6.1), which has been used for

DEFINITION: *Two pairs* $(x, y)$ *and* $(x', y')$ *in the set* $S \times S$ *are called equivalent if and only if*:

$$xy' = x'y \qquad (R_2)$$

If two elements in $S \times S$ satisfy the equivalence relation $(R_2)$, we write briefly:

$$(x, y) \equiv (x', y') \pmod{R_2}$$

This equivalence relation is reflexive, symmetric and transitive.

The equivalence relation $(R_2)$ divides the elements of the set $S \times S$ up into "disjoint classes" (*partitioning* of the set $S \times S$). Each of these equivalence classes is characterized by any one pair contained in it. For, if $(x, y)$ is a given pair in $S \times S$, then all elements of the class represented by $[x, y]$ are obtained by seeking all pairs equivalent to $(x, y)$, the class containing $(x, y)$ being denoted by $[x, y]$. The set of the classes into which the elements of $S \times S$ are partitioned is called the quotient set with respect to $(R_2)$; it is written $(S \times S)/R_2$.

A multiplication can be defined between the classes into which $S \times S$ is partitioned by the equivalence relation $(R_2)$, that is, between the elements of the quotient set $(S \times S)/R_2$:

DEFINITION: *The product of two classes* $[x, y]$ *and* $[x', y']$ *is the class* $[xx', yy']$:

$$[x, y][x', y'] = [xx', yy'] \qquad (M)$$

This operation is uniquely determined and its result is independent of the pair that is picked out as a representative of the class under consideration. Indeed, if we have:

$$(x_1, y_1) \equiv (x, y) \pmod{R_2} \quad \text{and} \quad (x_1', y_1') \equiv (x', y') \pmod{R_2} \qquad (5.1)$$

then, according to the definition of the product of two classes, the following is also valid:

$$[x, y][x', y'] = [xx', yy'] \quad \text{and} \quad [x_1, y_1][x_1', y_1'] = [x_1 x_1', y_1 y_1']$$

We may also write:

$$(x_1 x_1', y_1 y_1') \equiv (xx', yy') \pmod{R_2} \qquad (5.2)$$

for, by virtue of the equivalence relations (5.1), it follows that:

$$x_1 y = xy_1 \quad \text{and} \quad x_1' y' = x' y_1'$$



By multiplication of these two equations we obtain:

$$x_1 x_1' yy' = xx' y_1 y_1'$$

from which (5.2) follows. This demonstrates that the product of two classes yields another class, that is, that the set of the classes is closed under multiplication. That this multiplication of classes is *associative* and *commutative* can immediately be verified by calculation.

There exists a *neutral element* in the set of the classes: this is the class $[x, x]$. Indeed, if $[y, z]$ is any one class and if this class is combined with the class $[x, x]$ in a multiplicative way, we obtain:

$$[y, z][x, x] = [x, x][y, z] = [xy, xz]$$

The last-mentioned class is the same as the class $[y, z]$ since:

$$(xy, xz) \equiv (y, z) \pmod{R_2}$$

In the set of the classes there exists an *inverse* of each element. If $[x, y]$ is any one class, then its inverse is the class $[y, x]$, for we may write:

$$[x, y][y, x] = [xy, xy]$$

and the latter class is the neutral element by virtue of $(xy, xy) \equiv (x, x) \pmod{R_2}$.

With this the following theorem has been proved:

THEOREM I: *The classes of pairs of V-elements, that is, the elements of the quotient set* $(S \times S)/R_2$, *form an Abelian group G with respect to the operation* $(M)$.

Whereas the elements of $S$ are V-elements, the elements of $G$ are "classes of pairs of V-elements".

Each element of a certain proper subset $G'$ of $G$ can be related uniquely to an element of $S$, and *vice versa*. If we have $x \in S$, then the mapping:

$$x \to [xy, y] \qquad y \in S$$

relates just one class – that is, an element of $G' \subset G$ – to each element $x \in S$. For if $y \in S$ and $z \in S$, then $[xy, y] = [xz, z]$ because $(xy, y) \equiv (xz, z) \pmod{R_2}$. Different elements of $S$ correspond to different elements of $G'$. Indeed, if $x' \neq x$ and if it is true that:

$$x \to [xy, y] \quad \text{and} \quad x' \to [x'z, z]$$

then we should obtain $(xy, y) \equiv (x'z, z) \pmod{R_2}$ or $x(yz) = x'(yz)$ *if* generating vector space of a free system, these vector spaces will not be disjoint if and only if $\mu = \mu'$. Hence, $x$ and $x'$ must belong to the same vector space $A^{\nu+\mu}$. Furthermore, (4.1) requires the equality $\alpha\xi = \alpha\xi'$. It follows then from $\alpha \neq 0$ that $\xi = \xi'$, and, therefore, that $x = x'$.

As every semi-group whose elements are regular is called a "*regular semi-group*", $S$ is a "*commutative regular semi-group*".

What about *division?* When dividing we ask for an element $x \in S$ which is the solution of the equation:

$$a^{(\mu)}x = a^{(\nu)} \qquad a^{(\mu)}, a^{(\nu)} \in S \tag{4.2}$$

If this equation is to have any solution, it cannot possibly have more than one. If there were two solutions $x, x' \in S$, it would follow that:

$$a^{(\mu)}x = a^{(\nu)} = a^{(\mu)}x'$$

But since the elements of $S$ are regular, we have $x = x'$.

However, division in $S$ is only possible under restrictions, namely, if and only if $\mu < \nu$.

Indeed, if $\mu < \nu$ and $a^{(\mu)} = \alpha a_0^{\mu}$, $a^{(\nu)} = \alpha' a_0^{\nu}$, $x = \xi a_0^{\lambda}$, it follows from substitution into (4.2):

$$(\alpha\xi)a_0^{\mu+\lambda} = \alpha' a_0^{\nu}$$

The elements at both sides must belong to the same vector space. Hence, $\mu+\lambda = \nu$. The solution of the last-mentioned equation will only be a natural number $\lambda$, if $\mu < \nu$ so that $\lambda = \nu-\mu$.

On the other hand, division is not feasible if $\mu \geqq \nu$, for in this case the solution of the equation $\mu+\lambda = \nu$ cannot be a natural number. This shows that division in $S$ is only feasible under restrictions but not generally.

Now, in dimensional analysis symbols like, for instance, $(\text{metre})^{-1}$, $(\text{metre})^{-2}$ and $(\text{second})^{-1}$ are used without a definition of what a "unit" like $(\text{metre})^{-1}$ means. Obviously a mathematical foundation is here required. Such a foundation will be presented in what follows.

## 5 Imbedding a Commutative Regular Semi-Group in a Group

It follows from the exposition thus far given that the semi-group $S$ contains "too few" elements for enabling division without restrictions. An attempt will therefore be made to extend from the regular semi-group $S$ to a group $G$ in a manner analogous to the steps leading from the set of integers to that



of rational numbers. Detailed treatments of this procedure are given by N. Bourbaki,[1] M. Queysanne and A. Delachet,[2] and P. Dubreil,[3] in their above-quoted works. In order to elucidate this procedure it may be recalled that a set $G$ is called a *group* if its elements have the following properties:

(a) the elements of $G$ form a semi-group;

(b) there exists a neutral element;

(c) to every element of $G$ there corresponds a unique inverse element of $G$.

If, furthermore, the commutative law applies, then $G$ is called a *commutative* or *Abelian group*.

It follows from these properties that all elements of a group are regular. Hence, every group is at the same time a regular semi-group. Moreover, it follows from the existence of the inverse element that in a group division is feasible without restrictions. These properties of the group make it possible to give symbols like $(\text{metre})^{-1}$ and $(\text{second})^{-1}$ a meaning.

One gets from a commutative regular semi-group $S$ to an Abelian group $G$ by "symmetricizing" the commutative regular semi-group and at the same time "imbedding it in a group". For simplicity the elements of $S$ will now be written $x, y, \ldots$ (without specifying the vector space to which the $S$-element under consideration belongs).

One gets from $S$ to $G$ by considering the set of all ordered pairs of elements in $S$, that is, the set:

$$\{(x, y), (x', y'), \ldots\} \quad \text{with} \quad x, y, x', y', \ldots \in S$$

This set $G$ is called the "Cartesian product" of the set $S$ and itself; it is written $S \times S$. Now a multiplication of elements in the set $S \times S$ is defined:

DEFINITION: *The product of two pairs* $(x, y)$ *and* $(x', y')$ *in the set* $S \times S$ *is understood to mean the pair* $(xx', yy')$:

$$(x, y)(x', y') = (xx', yy')$$

In this operation the pairs are multiplied component by component.

The product of two elements in $S \times S$ is again an element of $S \times S$ (closure). This multiplication is clearly *associative* and *commutative*.

Now in the set $S \times S$ an "equivalence relation" is introduced:

[1] Bourbaki, chapter I, pp. 24 ff.

[2] Queysanne and Delachet, pp. 99 ff.

[3] Dubreil, pp. 224 ff.

vector space which is generally distinct from the spaces of the factors. If, for instance, $A$ is the vector space of the lengths of the one-dimensional Euclidean space, then $A^2$ is the vector space of the areas of the two-dimensional Euclidean space, $A^3$ the vector space of the volumes of the three-dimensional Euclidean space, etc. The vector spaces $A, A^2, A^3, \ldots$ are here disjoint. This means that a length (that is, an element of $A$) and an area (that is, an element of $A^2$) belong to distinct vector spaces and can, therefore, not be added.

Let $A$ still be the vector space of the length of the one-dimensional Euclidean vector space. If $B$ is, for instance, the vector space of the "time differences" – which is distinct from $A$ –, the foregoing argument can be generalized. Instead of $A^2$, that is, the vector space of "Length × Length", we now get the vector space of "Length × Time". The latter vector space, written as $AB$, is distinct both from $A$ and $B$. Consequently, operations of addition of elements of $AB$ and elements of $A$ or $B$ are impossible.

Generalization of the foregoing considerations leads to a system of $\rho$ vector spaces $A, B, \ldots, L$, where $\rho$ is a natural number, together with the set:

$$N = \{A^\alpha B^\beta \ldots L^\lambda \mid \alpha \in \mathbb{N}, \beta \in \mathbb{N}, \cdots\}$$

where $\alpha, \beta, \ldots, \lambda$ are natural numbers. Here, $N$ denotes the set of vector spaces generated by the system $A, B, \ldots, L$. Of all systems of $\rho$ vector spaces those are to be given special notice which generate a set $N$ of vector spaces that contains only vector spaces which are pairwise disjoint.

DEFINITION: *A system consisting of $\rho$ vector spaces $A, B, \ldots, L$ is called a free system of rank (or of order) $\rho$, if the elements of $N$ are vector spaces that are pairwise disjoint; if this is not so, the system is called a bound one. The $\rho$ vector spaces $A, B, \ldots, L$ are designated as the generating vector spaces of the free system.*

Let $A, B, \ldots, L$ be a free system of rank $\rho$;
let $\xi a_0^\alpha b_0^\beta \ldots l_0^\lambda$ be an element of the vector space $A^\alpha B^\beta \ldots L^\lambda$;
let $\xi' a_0^{\alpha'} b_0^{\beta'} \ldots l_0^{\lambda'}$ be an element of the vector space $A^{\alpha'} B^{\beta'} \ldots L^{\lambda'}$.
*If*:

$$\xi a_0^\alpha b_0^\beta \ldots l_0^\lambda \equiv \xi' a_0^{\alpha'} b_0^{\beta'} \ldots l_0^{\lambda'} \pmod{R_1},$$

*then*:

$$(\alpha, \beta, \ldots, \lambda) = (\alpha', \beta', \ldots, \lambda')$$

For, if the two $\rho$-tuples were different, the two elements under considera-



tion would belong to distinct vector spaces, that is, they could not possibly be $R_1$-equivalent.

If physicists are concerned with *third order, fourth order* or *fifth order systems*, they mean *free systems* of order 3, 4 or 5. A system of basic elements $a_0, b_0, \ldots, l_0$ of the above-mentioned vector spaces $A, B, \ldots, L$ that are the generating vector spaces of the free system is called a system of "basic units" in physical literature.[1]

## 4 Free Systems with One Generating Vector Space

For the sake of simplicity, the case of a free system of rank one may be considered first. Let $A$ be the generating vector space. The set $N$ then consists of the vector spaces $A, A^2, \ldots$. Since $A$ is the generating vector space of a free system, these vector spaces are pairwise disjoint, according to the definition of a free system.

Let us form the union of the vector spaces $A, A^2, \ldots$ and remove the zero elements of $A, A^2, \ldots$. This is done in order to enable the division of V-elements, that will be discussed in what follows, without restrictions. The set of elements of vector spaces thus obtained will be labelled $S$. The elements of $S$ are then the non-zero elements of the vector spaces $A, A^2, \ldots$.

Since multiplication of any two elements of $S$ yields another element of $S$ (closure), and since the associative and commutative laws apply, the elements of $S$, too, form a *commutative semi-group*.

THEOREM: *The elements of $S$ are "regular", that is, it follows from*:

$$a^{(\nu)}x = a^{(\nu)}x', \quad a^{(\nu)} = \alpha a_0^\nu \quad \textit{with}$$
$$a^{(\nu)} \in A^\nu \quad \text{and} \quad a^{(\nu)}, x, x' \in S$$

*that* $x = x'$.

Indeed, if $x = a^{(\mu)} = \xi a_0^\mu \in S$ and $x' = a^{(\mu')} = \xi' a_0^{\mu'} \in S$, the following equation will be valid:

$$\alpha \xi a_0^{\nu+\mu} = \alpha \xi' a_0^{\nu+\mu'} \tag{4.1}$$

where the product at the left hand side belongs to the vector space $A^{\nu+\mu}$ and the product at the right to the vector space $A^{\nu+\mu'}$. Equation (4.1) requires, therefore, that these two vector spaces are not disjoint. Since $A$ is the

[1] Two German expressions introduced by R. Fleischmann in his article: *Einheiteninvariante Größengleichungen, Dimensionen*, are: *Basisgrößenarten* and *Ausgangsgrößenarten*.

*The operation of addition of elements belonging to distinct vector spaces is excluded.*

It follows from this property of the algebraic structure of dimensional analysis that *the elements of the V-set do not constitute a ring.*

On the other hand, the operation of multiplication and division of V-elements does occur. We will first consider the multiplication of V-elements. Let $A$ and $B$ be two vector spaces which are not necessarily distinct, and assume: $a \in A$, $b \in B$.

(4) Let an *operation of multiplication* of any pair of elements, $a \in A$ and $b \in B$, be defined; let this operation associate just one element – to be designated as $ab$ – with the pair of elements $(a, b)$. The element $ab$ is called the *product of $a$ and $b$.*

Let this operation of multiplication have the following properties:

(4.1) If $a \in A$, $b \in B$ and $c \in C$, then:
$(ab)c = a(bc)$ (associative law);

(4.2) $ab = ba$ (commutative law);

(4.3) $(\alpha a)b = \alpha(ab)$ (associative law under multiplication by an element of $\Omega$);

(4.4) The set of all products $ab$ where $a \in A$ and $b \in B$ constitutes a one-dimensional vector space over the field $\Omega$; let us designate it as $AB$ (stability of the product);

(4.5) If $a$ differs from the zero element of $A$ and $b$ from the zero element of $B$, then the product $ab$ is different from the zero element of $AB$ (no divisors of zero).

The following conclusions can be drawn from the properties (4):

(a) If $a \in A$ and $b_1$ and $b_2 \in B$, we have:

$+_B, +_{AB}$

$$a(b_1+b_2) = ab_1+ab_2$$

For if $b_0$ is a basis of $B$ and if $b_1 = \beta_1 b_0$ and $b_2 = \beta_2 b_0$, it follows from the distributive law (2.2) that:

$$\begin{aligned} a(b_1+b_2) &= a(\beta_1 b_0+\beta_2 b_0) \overset{2.2}{=} a\{(\beta_1+\beta_2)b_0\} = \\ &\overset{4.2}{=} \{(\beta_1+\beta_2)b_0\}a \overset{4.3}{=} (\beta_1+\beta_2)(b_0 a) = \\ &\overset{2.2}{=} \beta_1(b_0 a)+\beta_2(b_0 a) \overset{4.3}{=} (\beta_1 b_0)a+(\beta_2 b_0)a = \\ &\overset{4.2}{=} a(\beta_1 b_0)+a(\beta_2 b_0) = ab_1+ab_2 \end{aligned}$$

(b) Let $a_0$ be a basis of $A$ and $b_0$ a basis of $B$. We can than write:

$$a = \alpha a_0 \quad \text{and} \quad b = \beta b_0$$



It follows from the properties (4) and (2) that:

$$\begin{aligned} ab &= (\alpha a_0)(\beta b_0) = \alpha\{a_0(\beta b_0)\} = \alpha\{(\beta b_0)a_0\} = \\ &= \alpha\{\beta(b_0 a_0)\} = (\alpha\beta)(b_0 a_0) = (\alpha\beta)(a_0 b_0) \end{aligned}$$

According to (4.5), $a_0 b_0$ is distinct from the zero element of $AB$ and may, therefore, be chosen as a basis of the vector space $AB$. The co-ordinate of $ab$ with reference to the basis $a_0 b_0$ is $\alpha\beta = \gamma$. If we let $\gamma$ pass through all elements of $\Omega$, we obtain all elements of the vector space $AB$. By reason of the invariance of the representation the elements of $AB$ that are so generated are independent of the choice of the basis $a_0 b_0$.

(c) *The V-elements constitute a commutative semi-group*, that is, the set of the V-elements has the following properties:

comm. semi group

(a′) The product of two V-elements is a V-element (closure);

(b′) The multiplication is associative;

(c′) The multiplication is commutative.

(d) It can be shown by means of induction that the result of multiplying more than three factors is independent of the way in which the factors are associated. Moreover, according to the commutative law the factors may be interchanged arbitrarily without the value of the product being affected.

Let $a_1, a_2, \ldots, a_\mu$ be elements of the vector space $A$. If $a_\lambda = \alpha_\lambda a_0$ where $a_0$ is a basis of $A$, it follows from property (4) that:

$$\prod_{\lambda=1}^{\mu} a_\lambda = \Big(\prod_{\lambda=1}^{\mu} \alpha_\lambda\Big) a_0^\mu$$

The product $\prod_{\lambda=1}^{\mu} a_\lambda$ is an element of a vector space written as $A^\mu$; $a_0^\mu$ is a basis of this vector space. If $a \in A$ where $a = \alpha a_0$, then $a^\mu = \alpha^\mu a_0^\mu$ which is also an element of the vector space $A^\mu$. The elements of the vector space $A^\mu$ will occasionally be written as $a^{(\mu)}$.

If $a_1, a_2, \ldots, a_\mu$ are elements of the vector space $A$ and $b_1, b_2, \ldots, b_\nu$ are elements of the vector space $B$, the product:

$$\Big(\prod_{\kappa=1}^{\mu} a_\kappa\Big)\Big(\prod_{\lambda=1}^{\nu} b_\lambda\Big)$$

is an element of the vector space $A^\mu B^\nu$. If $a_0$ is a basis of $A$ and $b_0$ of $B$, then $a_0^\mu b_0^\nu$ is a basis of the vector space $A^\mu B^\nu$.

Generally the V-element $ab \in AB$ will be neither an element of $A$ nor of $B$. This means that the operation of multiplying two V-elements generates a result false? It is the misleading assumption that 273.16 °$K$ and 0 °$C$ would be V-elements. But in fact they are "points" on a scale of temperatures. It should be obvious from the example that multiplication of "points" by elements of $\Omega$ does not make sense. Only "*temperature differences*" can be elements of the vector space "Temperature". These differences must be measured in terms of a unit called grd ("grade") or a multiple thereof.

There is still another reason for the "equation" (2.1) not being an equation in the sense of dimensional analysis. Neither °$K$ nor °$C$ are bases of the vector space Temperature, for out of any pair of bases of a vector space the one is obtained by multiplying the other by a number. Now, since grd is a basis of the vector space Temperature, any other basis must be a multiple of grd.

If $b$ denotes temperature measured from the "Kelvin zero-point" and $b'$ the same from the "Celsius zero-point", we have:

$$b - b' = 273.16 \text{ grd} \tag{2.2}$$

Both $b$ and $b'$ are here temperature points to be measured in terms of grd (or multiples thereof); hence, they are not V-elements. On the other hand, $b-b'$ is indeed a V-element, and (2.2) is an equation in such elements: it is, therefore, an "equation relating physical quantities".

The same is true in writing "points of time" (times of departure and arrival), "points of altitude" and "overpressures". For instance, the "equation":

$$13\ ^{\mathrm{h}}\mathrm{CET} = 12\ ^{\mathrm{h}}\mathrm{WET} \tag{2.3}$$

where $^{\mathrm{h}}$CET denotes Central-European Time and $^{\mathrm{h}}$WET Western-European Time, has the same character as (2.1). Multiplication of both sides by $\frac{1}{2}$ yields:

$$6^{\mathrm{h}}\ 30\ ^{\mathrm{m}}\mathrm{CET} = 6\ ^{\mathrm{h}}\mathrm{WET}$$

which is obviously a false result. A false statement is also obtained by adding the equation $2\ ^{\mathrm{h}}\mathrm{CET} = 1\ ^{\mathrm{h}}\mathrm{WET}$ (an equation that does make sense) to equation (2.3): as a result, we would obtain the equation $15\ ^{\mathrm{h}}\mathrm{CET} = 13\ ^{\mathrm{h}}\mathrm{WET}$ (which does not make sense). The reason for this is that (2.3) is a relation between "points of time", not "differences of time", and only the latter are V-elements.

It is always possible to come from "points" to "vectors" – that is, V-elements – by considering differences of points and then fixing a basis for the vector space to which these differences belong as elements. For instance, the travel time of a train (which is a V-element) can always be specified by taking the difference of the times of arrival and departure. This is a difference of points of time. The elements of the vector space "Time" are, therefore, "time differences". On the other hand, the addition of two points of time – say, time of departure *plus* time of arrival – does not make sense from the physical point of view.



Finally, let us recall how we measure on a straight line. As a unit of measurement we take a directed line segment $\overrightarrow{OE}$ (where $O$ and $E$ are two different points of the straight line). The length of this directed line segment may be 1 metre. If $\overrightarrow{OE}$ is carried from an arbitrary point $A$ of the straight line, say, through a distance of $v$ times the length of $\overrightarrow{OE}$ in the direction $OE$, and if then a point $B$ of the straight line is obtained, we write:

$$\overrightarrow{AB} = v\overrightarrow{OE} \tag{2.4}$$

where $v$ is a non-negative integer. If $\overrightarrow{OE}$ is carried in the reverse direction, that is, in the direction from $E$ to $O$, equation (2.4) remains valid, but $v$ is now a negative integer. The displacements of $\overrightarrow{OE}$ here required are translations with the length of $\overrightarrow{OE}$ being invariant. For the case where $v$ is not an integer the reader may be referred to the work of O. Hölder that has been cited in the Introduction.

Two directed line segments $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ lying on one straight line are "equivalent" in the sense discussed in the second paragraph of this section, if $\overrightarrow{AB}$ can be displaced along the straight line in such a way as to let $A$ coincide with $A'$ and $B$ with $B'$. Although the directed line segments $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ will generally contain different points of the straight line, they represent the same free vector, and, therefore, also the same V-element.

## 3 Multiplication of V-Elements

It has already been shown that in physical calculations elements of the same vector space are added (subtracted). Moreover, the elements of a vector space can be multiplied by numbers (elements of $\Omega$). On the other hand, it must be assumed that addition (subtraction) of V-elements belonging to distinct vector spaces is not allowed, since addition of, for instance, 2 metres and 3 seconds "does not make sense from the physical point of view". Let us, therefore, require the following axiom:

THEOREM II: *Two V-elements, x and y, belong to the same one-dimensional vector space if and only if there exists a pair of numbers,* $(\alpha, \beta) \neq (0, 0)$, *such that*

$$\alpha x = \beta y \qquad (R_1)$$

$x, y \in X \Rightarrow \exists \alpha, \beta$

PROOF: The condition is *necessary*. If $x = y$, then we can put $\alpha = \beta = 1$. If $x \neq y$, then one of the two elements, which both belong to a vector space $X$, differs from the zero element of $X$. If, for instance, $x$ is this element, then, according to Theorem I, $x$ can be chosen as basis of $X$ and from $y \in X$ and property (3) we obtain $y = \alpha x$. Hence, $(\alpha, \beta) = (\alpha, 1) \neq (0, 0)$.

The condition is *sufficient*. Take, for instance, $\beta \neq 0$. From $(R_1)$ one obtains: $y = (\alpha/\beta)x$. If $x$ is an element of the vector space $X$, it follows from this equation that $y \in X$ is true. This is independent from $x$ being identical with the zero element of $X$, or not.

Types, kinds, dims

In physics quantities belonging to the same vector space are called "of the same kind".

The relation $(R_1)$ can be understood to mean an "equivalence relation", since it provides a necessary and sufficient criterion for deciding whether or not two V-elements belong to the same vector space. If two V-elements, $x$ and $y$, belong to the same vector space, this is written as:

$$x \equiv y \qquad (\text{mod } R_1)$$

(which is read: $x$ and $y$ are equivalent modulo $R_1$). This equivalence relation is reflexive, symmetric, and transitive:

reflexive: $x \equiv x \ (\text{mod } R_1)$;

symmetric: if $x \equiv y \ (\text{mod } R_1)$, then $y \equiv x \ (\text{mod } R_1)$;

transitive: if $x \equiv y \ (\text{mod } R_1)$ and $y \equiv z \ (\text{mod } R_1)$, then $x \equiv z \ (\text{mod } R_1)$.

From the foregoing another theorem results:

THEOREM III: *Two V-elements are equal if and only if they belong to the same vector space and have the same co-ordinate with respect to some arbitrarily chosen basis.*

Therefore, any equation occurring in this calculus requires that both sides are elements of the same vector space.



## 2 Point Space and Vector Space

In physical literature one sometimes comes across "equations" not containing V-elements. In most cases we are then concerned with relations between elements of a "point space", not between elements of a vector space.

In considering the equations occurring in the axioms of the vector space it should be noted that the elements of a vector space are so-called "free vectors" (as opposed to "bound vectors"). A free vector is here understood to mean a class of directed segments of line (point differences) where each element of the class arises by translation of a given element. Two directed line segments (point differences) belonging to the same class – that is, which can be made to cover each other by a translation – are called *equivalent* (we are here concerned with a kind of equivalence that differs from the one treated in section 1). Usually one contents himself with characterizing a class by indicating one arbitrary element belonging to it (the other elements arise from the given element by translation). The set of all directed line segments (point differences) is divided up into *disjoint classes* by the equivalence relation.[1] Finally, it should be noted that the translations constitute an additive Abelian group, the neutral element (or identity) of which is the translation through a distance zero. For further details of the connection between point space and vector space the reader may be referred to the works of, for instance, E. Sperner [2] and A. Duschek and A. Hochrainer.[3]

Let us consider as an example the following "equation" taken from physics:

$$273.16\,^\circ K = 0\,^\circ C \qquad (2.1)$$

where $^\circ K$ denotes Kelvin degrees and $^\circ C$ Celsius degrees (centigrade). This is not an equation in the sense of dimensional analysis: as a matter of fact, it represents a relation between *points*, not between V-elements. Indeed, if it were an equation in V-elements, it would be possible to multiply both members of the equation by the same number different from zero – for instance, the number 2 –, that is, an element of $\Omega$, without the validity of the equation being affected. However, it would then become $546.32\,^\circ K = 0\,^\circ C$, which is inconsistent with the original equation (2.1). Why is this

[1] See on this: M. QUEYSANNE and A. DELACHET, *L'algèbre moderne*, Paris, 1955, p. 85.
[2] Sperner, Vol. II, pp. 25–26.
[3] A. DUSCHEK and A. HOCHRAINER, *Grundzüge der Tensorrechnung in analytischer Darstellung*, Vol. I, fourth edition, Vienna, 1960, pp. 16–17.

laws this algebraic structure obeys are the same as those for calculations with the well-known vectors of two- or three-dimensional space. In connection with dimensional analysis the groups with operators were mentioned for the first time, but not used, by M. Landolt.[1]

An important property of the vector space is the following: *if $\alpha a = o$, then $\alpha = 0$ or $a = o$, or $\alpha = 0$ and $a = o$.*

This may be demonstrated as follows: If $\alpha \neq 0$, then $\beta = 1/\alpha \neq 0$; now, multiplication of $\alpha a = o$ by $\beta$ gives:

$$\beta(\alpha a) = a = \beta o = \beta(a-a) = \beta a - \beta a; \quad \text{hence, } a = o.$$

If $a \neq o$, then $\alpha = 0$; for, if $\alpha \neq 0$ we would obtain $a = o$, which has been proved above, but this is inconsistent with the assumption $a \neq o$.

Finally, let it be required that the elements of $A$ have the following properties:

(3) *There is at least one element $a_0$ in $A$ which is different from zero, and with any element $a$ of $A$ there is just one number $\alpha$ such that*

$$a = \alpha a_0$$

If the elements of $A$ have the property (3) in addition to the properties (1) and (2), it is said that the elements of $A$ constitute a "*one-dimensional vector space over the field $\Omega$*".

The element $a_0$, which is usually kept fixed, is called "the basis of the one-dimensional vector space $A$" (in physics: the unit measuring the quantities belonging to $A$). The number $\alpha \in \Omega$ associated with the element $a \in A$ after $a_0$ has been chosen is designated as the "co-ordinate" of $a$ with reference to the basis $a_0$ (in physics: the number measuring the quantity $a$ in terms of the unit $a_0$). Hence, $\alpha = 0$ if and only if $a = o$.

The following theorem results from the properties (1)–(3) of the one-dimensional vector space $A$:

THEOREM I: *Any element $a$ of the one-dimensional vector space $A$ which differs from the zero element $o$ of $A$ can serve as basis of the vector space $A$.*

In effect, if $a_0 \neq o$ is chosen as basis of $A$, then according to property (3) every element $a \in A$ can be written as $a = \alpha a_0$ (where $\alpha \in \Omega$). Now, if $a'_0$ is an element of $A$ different from both $o$ and $a_0$, then $a'_0$ can be written as:

$$a'_0 = \beta a_0 \qquad (\beta \neq 0, \quad \beta \neq 1)$$

[1] Landolt, p. 8.



Since it follows from this equation that $a_0 = (1/\beta)a'_0$, every element $a \in A$ can be written as:

$$a = \frac{\alpha}{\beta} a'_0 = \alpha' a'_0 \qquad \text{Q.E.D.}$$

For the sake of completeness the formulae describing the *transformation of co-ordinates as a consequence of changing the basis* of a vector space may now be given. If a V-element $a \in A$ is described with the aid of two different bases $a_0$ and $a'_0$:

$$a = \alpha a_0 \tag{1.1}$$

$$a = \alpha' a'_0 \tag{1.1'}$$

then two generally different co-ordinates $\alpha$ and $\alpha'$ are associated with one and the same element $a \in A$.

If $a'_0 = \beta a_0$ (where $\beta \neq 0$, $\beta \neq 1$), then these two co-ordinates are related by one of the two relations following from (1.1) and (1.1'):

$$\alpha = \alpha' \beta \tag{1.2}$$

$$\alpha' = \frac{\alpha}{\beta} \tag{1.2'}$$

(1.2) and (1.2') serve as transformation formulae for the co-ordinates when the basis is changed. The elements themselves are independent of the choice of the basis, or, in other words, they are "invariant" with respect to a change of basis. In physics an extensive literature exists on this property of V-elements (labelled quantities in physics); see, for instance, the above-mentioned work of J. Wallot.

Apart from the elements of an $A$-space, in the observations which follow elements of other one-dimensional vector spaces will be considered, say, the elements of an $A$-space, a $B$-space, a $C$-space, . . .. Let the set of these spaces have the following property:

*There is a countably infinite number of one-dimensional vector spaces which are pairwise disjoint.*

*Let the set of elements of all these vector spaces be called the set of V-elements, or briefly the "V-set".*

Whether or not two elements of the V-set belong to one and the same vector space can be determined by means of the following theorem.

analysis may be compared with a computer which processes the data it is given in a way determined by its structure, ignoring those properties for which it lacks an "organ". If certain results of dimensional analysis should appear surprising at first sight, this is due to the fact that it is impossible for this computer to allow for special features alien to dimensional analysis. For instance, dimensional analysis cannot possibly allow for differences between tensors of different rank.

## 1 Vector Spaces of V-Elements

Let $\Omega$ be the field (the set) of the real or the complex numbers. Denote the elements of $\Omega$ as:

$$\alpha, \beta, \gamma, \ldots \qquad \Omega = \mathbb{R} | \mathbb{C}$$

Further, let $A$ be an infinite set, called "$A$-space". Its elements will be called "V-*elements* of the $A$-space" and will be written as:

$$a, a_1, a_2, \ldots$$

*Let the sets $A$ and $\Omega$ have no elements in common.* This requirement means that no element of the $A$-space is a number. For instance, the V-element "3 metres" and the number 3 are to be considered as distinct from each other. In section 7 more will be said about the possibility of "identifying" certain V-elements with numbers.

Let the set $A$ have the following properties:

(1) An operation combining elements of $A$, called *addition*, is defined. This operation associates any pair of elements $a_1$ and $a_2$ of $A$ with just one element $a_3$ of $A$:

$$a_1 + a_2 = a_3$$

The element $a_3$ is called the "sum" of $a_1$ and $a_2$. Since the combination of two elements of $A$ yields another element of $A$, it is said that we are here concerned with an "inner composition" of elements of $A$. (Addition is an operation that is called "closed" in $A$.)

Let this operation have the following properties:

(1.1) $a_1 + (a_2 + a_3) = (a_1 + a_2) + a_3$ (associative law);

(1.2) $a_1 + a_2 = a_2 + a_1$ (commutative law);

(1.3) There exists just one "neutral element" or zero element, written



as $o$, so that for any element $a \in A$:

$$a + o = a$$

(1.4) Any element $a \in A$ has a unique "inverse". The latter element is written as $-a$:

$$a + (-a) = o$$

If the elements of a set have the properties under (1), the set is called an "additive Abelian group". It follows from the properties (1) that subtraction of two elements $a_1$ and $a_2$ of $A$ is possible in a unique way and without restriction. The difference $a_2 - a_1$ is the element $a_3$ for which $a_1 + a_3 = a_2$. So we have for this element:

$$a_3 = a_2 + (-a_1)$$

(2) A multiplicative operation combining elements of $\Omega$ (numbers) and elements of $A$ is defined. This operation associates any element $\alpha \in \Omega$ and any element $a \in A$ with just one element $a_1 \in A$:

$$a_1 = \alpha a = a\alpha$$

This is a case of "outer composition", since the combination of an element of $A$ and an element of $\Omega$ not belonging to $A$ yields an element of $A$.

Let this combination of elements have the following properties:

(2.1) $\alpha(\beta a) = (\alpha\beta)a$ (associative law);

(2.2) $(\alpha+\beta)a = \alpha a + \beta a$ (distributive law for addition of $\Omega$-elements);

(2.3) $\alpha(a_1 + a_2) = \alpha a_1 + \alpha a_2$ (distributive law for addition of $A$-elements);

(2.4) $1a = a$ (unitary $\Omega$-module).

If the elements of a set have the properties specified under (1) and (2), the elements are said to constitute a "*vector space over the field* $\Omega$".

As usual in algebra, a "vector space" is here understood to mean a special algebraic structure belonging to the "groups with operators" (*groupes à opérateurs*). The elements of $\Omega$, that is, the numbers, are then called "operators"; the group under consideration is the additive Abelian group defined under (1). The above-quoted work of N. Bourbaki enters into such groups in detail.[1] The name vector space originates from the fact that the

[1] Bourbaki, chapter I, pp. 79 ff. and chapter II, pp. 1 ff.

Of fundamental importance for this "dimensional analysis" is the question of what operational rules apply to it. An axiomatic treatment of calculating and measuring in geometry has already been presented by O. Hölder.[1] The following discussion has a slightly different direction: it aims at describing the *algebraic structure* of dimensional analysis. That the algebraic method is more appropriate to the nature of the subject also appears from its extensive application in a number of recent publications; see, for instance, the works of M. Landolt, S. Drobot,[2] and R. Fleischmann.[3, 4, 5] As the terminology of algebra differs from that of physics, in the present text the names that are conventional in physics will be given concurrently with those used in algebra so as to make the argument more readily understood. The notation in what follows is also different from that prevailing in physics. It has been chosen only with regard to the properties of the structure.

There are two algebraic structures which are important in dimensional analysis, namely, the concepts of *vector space* and *group*. Both concepts have long been in use, occasionally without their being specially mentioned. For a long time, several authors – for instance, J. Wallot – have been advocating an "invariant notation" of "equations that relate physical quantities".[6] Behind this requirement is the concept of vector space. Evidence for the importance of the concept of group has been given by M. Landolt and R. Fleischmann in their above-mentioned publications. This paper has, of course, many points of contact with the work of other authors. It differs, however, from these in aiming mainly at analyzing the calculus and at giving an axiomatic treatment of dimensional analysis. An exposition of the algebraic concepts and theorems will be found in the works of N.

[1] O. Hölder, "Die Axiome der Quantität und die Lehre vom Maß", *Berichte der Sächsischen Akademie*, LIII (1901), pp. 1–64.

[2] S. Drobot, "On the Foundations of Dimensional Analysis", *Studia Mathematica*, XIV (1953), pp. 84–99.

[3] R. Fleischmann, "Die Struktur des physikalischen Begriffssystems", *Zeitschrift für Physik*, CXXIX (1951), pp. 377–400.

[4] R. Fleischmann, "Das physikalische Begriffssystem als mehrdimensionales Punktgitter", *Zeitschrift für Physik*, CXXXVIII (1954), pp. 301–308.

[5] R. Fleischmann, "Einheiteninvariante Größengleichungen, Dimensionen", *Der mathematische und naturwissenschaftliche Unterricht*, XII (1959–60), pp. 385–399 and 443–458.

[6] *Translator's Note*: This "invariant notation" implies, in the words of Bridgman (p. 37), "that the functional relation is of such a form that it remains true formally without any change in the form of the function when the size of the fundamental units is changed in any way whatever".



Bourbaki,[1] A. Chatelet,[2] P. Dubreil,[3] E. Sperner,[4] and B. L. van der Waerden.[5, 6] For the reader who is less familiar with modern algebra, a fuller exposition has been given than would be required from a strictly algebraic point of view.

The application of dimensional analysis is only demonstrated by means of a few elementary examples taken from geometry and mechanics. A treatment of the problem of the application of dimensional analysis to physics as a whole has not been attempted, because of differences in opinion concerning the interpretation of physical equations.

Since a generally accepted definition of the concept "physical quantity" is lacking, we shall avoid using the expression "physical quantity" in order to prevent misunderstanding. Instead, we introduce a new concept, named "V-element". This name has been chosen to emphasize that we are here concerned with elements of (abstract) vector spaces. As usual in axiomatics, the new concept is defined in an implicit way by enumerating the properties of the algebraic structure of dimensional analysis. Consequently, any quantity having the properties as required below may be called a V-element. The reader who has no objection to the idea can, therefore, understand a V-element to be a physical quantity.[7]

In describing the algebraic structure of dimensional analysis the limits imposed on this kind of analysis as such become apparent. Dimensional

[1] N. Bourbaki, *Eléments de Mathématique*, Vol. II, *Algèbre*, second edition, Paris, 1951–55, chapters 1 and 2.

[2] A. Chatelet, *Arithmétique et Algèbre modernes*, Vol. I, Paris, 1954.

[3] P. Dubreil, *Algèbre*, Vol I, second edition, Paris, 1954.

[4] E. Sperner, *Einführung in die analytische Geometrie und Algebra*, Vols. I and II, third and fourth editions, Göttingen, 1959.

[5] B. L. van der Waerden, *Algebra*, Vol. I, fourth edition, Berlin, Göttingen and Heidelberg, 1955.

[6] *Translator's Note*: For a brief but lucid exposition in English the reader may be referred to: R. G. D. Allen, *Basic Mathematics*, London & C., 1962, chapters 4, 6, 7 and 13; some formal development of sets, groups, rings, fields and vector spaces is given in section 3 of chapter 15. A more thorough introduction written for those who are meeting modern algebra for the first time is: E. M. Patterson and D. E. Rutherford, *Elementary Abstract Algebra*, Edinburgh, London and New York, 1965 (viii+211 pp.). The latter book does not, however, include set theory; this will be found in chapter 4 of Allen's book.

[7] *Translator's Note*: We can equally well understand a V-element to be an *economic* quantity. That is why Professor Quade's paper is appended to this book. Consequently, the remarks Professor Quade makes about physical calculations can also be regarded as valid for economic calculations.

Since $e$ is an abstract number, we have in (2.7.4) and (2.7.5):

$$y \in [1] \qquad (2.7.16)$$

$$q \in [1] \qquad (2.7.17)$$

so that we have abstract numbers only.

Moreover, we have in (2.7.6): [1]

$$Ey \quad \text{or} \quad d \ln y \in [1] \qquad (2.7.18)$$

$$Eq \quad \text{or} \quad d \ln q \in [1] \qquad (2.7.19)$$

and, therefore:

$$\frac{Ey}{Eq} \in [1] \qquad (2.7.20)$$

so that this elasticity expression cannot possibly have dimensions.

(2) It is, of course, entirely certain that the formulae (2.7.13) and (2.7.14) do not represent Samuelson's meaning in a correct way: the same may be true of (2.7.16) and (2.7.17). I think he will accept (2.7.8) – or (2.7.11) – and (2.7.9). This means, however, that his equations (2.7.4) and (2.7.5) are not dimensionally homogeneous and, therefore, meaningless from the viewpoint of economic theory. We may repair this imperfection by writing, for instance:

$$y \quad \text{or} \quad x_{(t)} = x_0 \cdot e^{\alpha t} \qquad (2.7.4 \text{ bis})$$

$$q \quad \text{or} \quad p_{(t)} = p_0 \cdot e^{\beta t} \qquad (2.7.5 \text{ bis})$$

which is consistent with formulae (2.7.8) – or (2.7.11) – and (2.7.9). We shall then find that $d \ln x_{(t)}/d \ln p_{(t)}$ or $Ey/Eq$ is an abstract number; but (2.7.6) will not hold good.

[1] See the preceding footnote.

MATHEMATICAL APPENDIX

# THE ALGEBRAIC STRUCTURE OF DIMENSIONAL ANALYSIS

WILHELM QUADE

*Translated from the German * by Frits J. de Jong*

## Summary

In describing the algebraic structure of dimensional analysis the concepts of *vector space* and *group* are found to be fundamental. Division of quantities is defined by embedding a commutative semi-group in an Abelian group. Of the groups homomorphic to this Abelian group a certain infinite Abelian group is of special importance. The properties of this group are described by using the additive notation. Investigation of functions of "dimensionless quantities" leads to a new proof of the $\pi$-Theorem of dimensional analysis (also called Buckingham's Theorem).

## Introduction [1]

In physics, particularly in quantitative physics, it is convenient to use a concept labelled "physical quantity", which usually occurs together with one called "equation relating physical quantities". Both concepts are used in the well-known calculus dealing with physical systems of measurement, which underlies dimensional analysis. This is a topic on which an extensive literature exists; for more detailed references the reader may consult the works of G. Birkhoff,[2] P. W. Bridgman,[3] M. Landolt,[4] G. Oberdorfer,[5] U. Stille,[6] and J. Wallot.[7]

* "Über die algebraische Struktur des Größenkalküls der Physik", *Abhandlungen der Braunschweigischen Wissenschaftlichen Gesellschaft*, XIII (1961), pp. 24–65.

[1] I am indebted to Professors R. Fleischmann, A. Hochrainer and U. Stille for many stimulating discussions. I thank my collaborators, Mr. H. Brass and Mr. St. Schottlaender, for their valuable criticisms made during the writing.

[2] G. Birkhoff, *Hydrodynamics*, Princeton, N. J., 1950.

[3] P. W. Bridgman, *Dimensional Analysis*, revised edition, New Haven and London, 1931.

[4] M. Landolt, *Größe, Maßzahl und Einheit*, second edition, Zürich, 1952.

[5] G. Oberdorfer, *Die Maßsysteme in Physik und Technik*, Vienna, 1956.

[6] U. Stille, *Messen und Rechnen in der Physik*, Brunswick, Germany, 1955.

[7] J. Wallot, *Größengleichungen, Einheiten, Dimensionen*, second edition, Leipzig, 1957.