Formal methods for accountable decision making

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- These slides: https://gitlab.pik-potsdam.de/botta/IdrisLibs/tree/master/lectures/2018-12-03.PIK.Global_change_management_week

Outline

- Decision problems in climate research
- A conceptual emission problem
- An environment for specifying and solving SDPs
- Specifying and solving the emission problem

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- Too rapid reductions may compromise the wealth of one or more upcoming generations but ...
- ... they may promote a transition to societies that are more wealthy, safe, fair and manageable.
- New technologies that significantly reduce the costs of very fast emission reductions may become available soon.

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- In this situation most countries have a free-ride opportunity!

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- If implemented, low emissions increase cumulated emissions less than high emissions.

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- Similarly, the probability that new technologies become available is low at the first decision step. It increases steeply after a critical number of steps.
- Once available, technologies stay available for ever.

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- The decision maker aim at maximising a sum of the benefits over all decision steps.

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  - What kind of solutions or advice can we offer to the decision maker?
  - What kind of guarantees can we provide for such solutions? Can we check that they are correct? What does this mean?

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  - More uncertainty about the implementability of decisions dictates earlier emission reductions.
  - More uncertainty about the implications of exceeding critical thresholds make earlier reductions sub-optimal.

Sequential decision problems: a visual sketch

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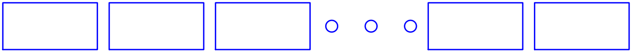
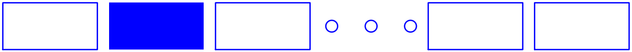
There are $n + 1$ decision steps to go ...



n+1 steps left

Sequential decision problems: a visual sketch

…here is the current state,



n+1 steps left

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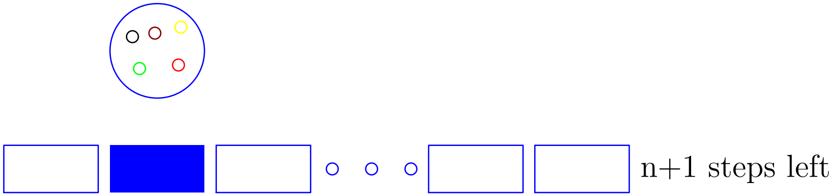
. . . here are your options.



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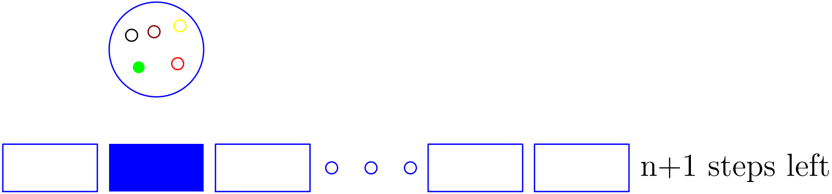
Pick one!



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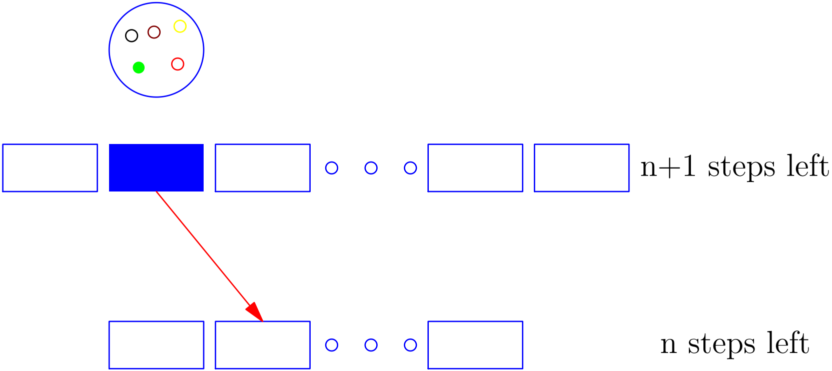
Move to a new state and . . .



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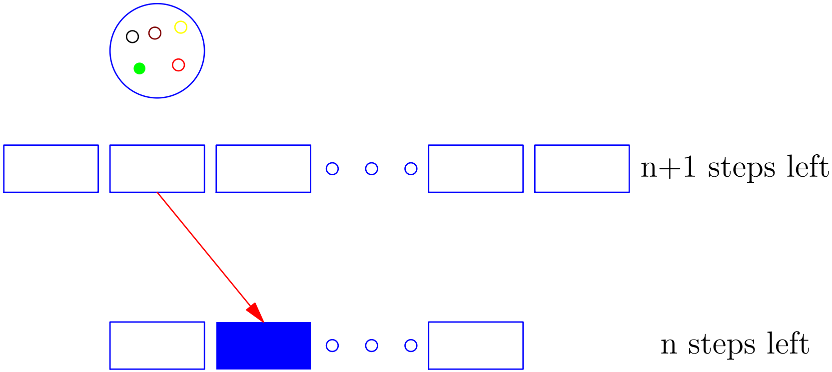
. . . collect rewards and face the next decision step!



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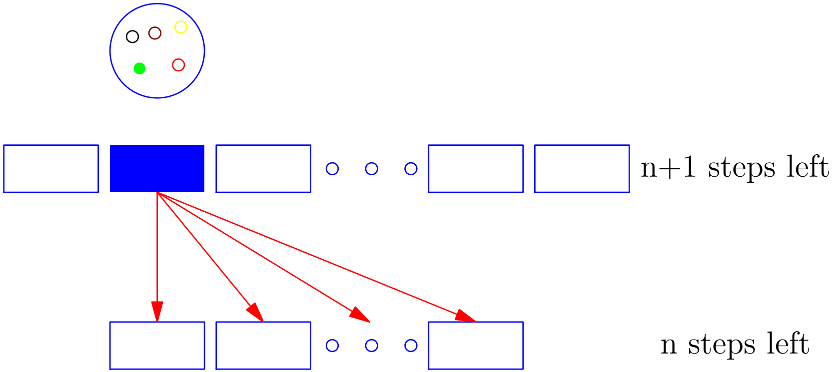

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- These implications justify calling $x = 1$ and $x = -1$ solutions of $x^2 = 1$!

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- This is a simple example of a problem specification.

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- Specifications can also be applied to clarify crucial notions. We have seen that in emission problems . . .

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- We can express this idea with a specification:

> *Let $S = \{High, Low\}$ and $p1, p2 : S \times S \to \mathbb{R}$ payoffs. A* **strategy profile** $(x, y) \in S \times S$ *is a* **Nash equilibrium** *iff* $\forall x', y' \in S$, $p1(x', y) \leqslant p1(x, y)$ *and* $p2(x, y') \leqslant p2(x, y)$.

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- Exercise: give a mathematical specification of the notion of optimality for policies.

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- In strongly typed languages, each valid expression has a type:

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\begin{aligned}
1 + 2 &: \mathbb{N} \\
\texttt{"Hello"} &: \mathit{String} \\
[1, 7, 3, 8] &: \mathit{List}\ \mathbb{N}
\end{aligned}
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- The judgment $e\ :\ t$ states that the expression e has type t.

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  denotes the application of the function f to the argument x.

  $$g \circ f$$

  denotes the composition of g and f. Functions can take functions as arguments:

  $$(\circ) : (b \to c) \to (a \to b) \to a \to c$$
  $$(g \circ f)\ x = g\ (f\ x)$$

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- In dependently typed languages, types can depend on values:

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\quad \mathit{Nil}\ :\ \mathit{Vect}\ Z\ a \\
\quad (::)\ :\ (x\ :\ a)\ \to\ (xs\ :\ \mathit{Vect}\ n\ a)\ \to\ \mathit{Vect}\ (S\ n)\ a
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- Thus, we can use types to encode specifications. For instance:

$$\begin{array}{l} \mathit{Injective}\ :\ (a\ \rightarrow\ b)\ \rightarrow\ \mathit{Type} \\ \mathit{Injective}\ \{a\}\ f = (x, y\ :\ a)\ \rightarrow\ f\ x = f\ y\ \rightarrow\ x = y \end{array}$$

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\quad (::)\ :\ (x\ :\ a)\ \rightarrow\ (\mathit{xs}\ :\ \mathit{Vect}\ n\ a)\ \rightarrow\ \mathit{Vect}\ (S\ n)\ a
\end{array}
$$

- Thus, we can use types to encode specifications. For instance:

$$
\begin{array}{l}
\mathit{Injective}\ :\ (a\ \rightarrow\ b)\ \rightarrow\ \mathit{Type} \\
\mathit{Injective}\ \{a\}\ f = (x, y\ :\ a)\ \rightarrow\ f\ x = f\ y\ \rightarrow\ x = y
\end{array}
$$

- This is almost a word-by-word translation of

$$f : A \rightarrow B \textit{ injective iff } \forall x, y \in A,\ f(x) = f(y) \Rightarrow x = y.$$

Dep. types and machine checkable specifications: wrap-up

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- 3: Implement *specP*.
- Example of 1:

$$\begin{aligned} &P &&: \mathbb{R} \to \mathbb{R} \\ &specP &&: (x : \mathbb{R}) \to 0 \leqslant x \to (P\ x) * (P\ x) = x \end{aligned}$$

An environment for specifying and solving SDPs

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- At decision step t, $Y\ t\ x$ denotes the controls available to the decision maker in $x\ :\ X\ t$:

$$Y \;:\; (t \;:\; \mathbb{N}) \;\to\; X\ t \;\to\; \mathit{Type}$$

An environment for specifying and solving SDPs

- At decision step t, $next\ t\ x\ y$ denotes the next states that can be reached by selecting control y in the current state x:

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- Notation:
  - $S\ t = t + 1$.
  - $M = Identity \Rightarrow$ no uncertainty, deterministic SDP
  - $M = List \Rightarrow$ non-deterministic SDP
  - $M = Prob \Rightarrow$ stochastic SDP

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- To specify a concrete SDP, one defines X, Y and *next*.
- For example, in the emission problem discussed in the beginning, $X\ t$ represents the cumulated emissions, the current emissions, availability of technologies for reducing emission and a state of the world.
- To complete the specification of a concrete SDP, we have to say which are the benefits that the decision maker wants to maximize:

$$\begin{aligned} &Val \ : \ Type \\ &reward \ : (t \ : \ \mathbb{N}) \ \rightarrow \ (x \ : \ X\ t) \ \rightarrow \ Y\ t\ x \ \rightarrow \ X\ (S\ t) \ \rightarrow \ Val \end{aligned}$$

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- The decision maker seeks controls that maximize a sum of the rewards, thus values of type *Val* have to be "addable":

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(\oplus) &: \mathit{Val} \rightarrow \mathit{Val} \rightarrow \mathit{Val} \\
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- In SDPs, *Val* is often $\mathbb{N}$ or $\mathbb{R}$.

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- Policies are functions from states to controls:

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- We can construct an empty policy sequence at every decision step.
- With a decision policy for step t and a sequence of n policies, we can construct a policy sequence for $n+1$ decision steps.

An environment for specifying and solving SDPs

- We can compute the value of taking n decisions according to a policy sequence in terms of a sum of the rewards obtained:

$$
\begin{array}{l}
val : (x : X\ t) \to PolicySeq\ t\ n \to Val \\
val\ x\ Nil = zero \\
val\ x\ (p :: ps) = reward\ t\ x\ y\ x' \oplus val\ x'\ ps\ \textbf{where} \\
\quad y\ \ : Y\ t\ x \\
\quad y\ \ = p\ x \\
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\end{aligned}
$$

- Remember that the decision maker seeks controls that maximize a sum of the rewards.
- *val* computes precisely such a sum for arbitrary policy sequences!

An environment for specifying and solving SDPs

- Thus, we can use *val* to express what it means for a policy sequence to be optimal:

$$
\begin{aligned}
&\mathit{OptPolicySeq} \;:\; \mathit{PolicySeq}\; t\; n \;\rightarrow\; \mathit{Type} \\
&\mathit{OptPolicySeq}\; ps = (x \;:\; X\; t) \;\rightarrow\; (ps' \;:\; \mathit{PolicySeq}\; t\; n) \;\rightarrow \\
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- ... How do we compute optimal policies?

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- The key idea lies in the notion of optimal extensions.

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- A policy p is an optimal extension of a policy sequence ps if there is no better way to make one more decision steps than $p :: ps$:

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&OptExt \;:\; PolicySeq\;(S\;t)\;m \;\rightarrow\; Policy\;t \;\rightarrow\; Type \\
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An environment for specifying and solving SDPs

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$$\begin{array}{lll} \mathit{Bellman} : & (\mathit{ps} \; : \; \mathit{PolicySeq} \; (S \; t) \; m) & \to \; \mathit{OptPolicySeq} \; \mathit{ps} \; \to \\ & (p \;\; : \; \mathit{Policy} \; t) & \to \; \mathit{OptExt} \; \mathit{ps} \; p \; \to \\ & \mathit{OptPolicySeq} \; (p :: \mathit{ps}) & \end{array}$$

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- Proving *Bellman* is not difficult but a little bit technical.

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- Instead of proving the result, we are going to apply it!

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- Then

$$\begin{aligned}
& backwardsInduction \;:\; (t \;:\; \mathbb{N}) \;\to\; (n \;:\; \mathbb{N}) \;\to\; PolicySeq\;t\;n \\
& backwardsInduction\;t\;Z \qquad = Nil \\
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computes optimal policy sequences for arbitrary SDPs!

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An environment for specifying and solving SDPs

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How do we apply the method?

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- Without lost of generality, we can take these increases to be zero and one.

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  - The availability of effective technologies for reducing emissions $T = \{\mathit{Available}, \mathit{Unavailable}\}$.
  - A state of the world $W = \{\mathit{Good}, \mathit{Bad}\}$.
- Thus, states are just tuples of 4 values:

$$X\ t = (\{0\,..\,t\}, E, T, W)$$

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- Once available, effective technologies stay available for ever.

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- Low/high emission decisions may or may not be implemented.
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- Exceeding *crE* may or may not turn the world into a bad state.

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Uncertainties

- Low/high emission decisions may or may not be implemented:
  - pLL – the probability of implementing low emissions when the current emissions measures are low and low emissions are chosen.
  - pLH – the probability of implementing low emissions when the current emissions measures are high and low emissions are chosen.
  - pHL – the probability of implementing high emissions when the current emissions measures are low and high emissions are chosen.
  - pHH – the probability of implementing high emissions when the current emissions measures are high and high emissions are chosen.
- Constraints: $pLH \leqslant pLL$ and $pHL \leqslant pHH$.

Uncertainties

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Uncertainties

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- Exceeding *crE* may or may not turn the world into a bad state:
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Transition function

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$$
\begin{array}{lcl}
\textit{prob}\,(e, \quad L, U, G) & = & pLH \;*\; (1 - pA1) \;*\; pS1 \\
\textit{prob}\,(e+1, H, U, G) & = & (1 - pLH) \;*\; (1 - pA1) \;*\; pS1 \\
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- Low emissions yield less benefits (more costs, less growth) than high emissions.
- Implementing low emissions when effective technologies are unavailable costs more than implementing emissions when these technologies are available.

Rewards

Without loss of generality, we can take the benefits of being in a good world for a step to be one and define

$$
\begin{aligned}
\mathit{reward}\ t\ x\ y\ (e, H, U, G) &= 1 + h \\
\mathit{reward}\ t\ x\ y\ (e, H, U, B) &= b + h \\
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$$

where $h, b, \mathit{lu}, \mathit{la} : \mathbb{R}$ fulfil $b \leqslant 1$, $0 \leqslant \mathit{lu} \leqslant \mathit{la} \leqslant h$.

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where $h, b, lu, la\ :\ \mathbb{R}$ fulfil $b \leqslant 1$, $0 \leqslant lu \leqslant la \leqslant h$. Notice that

- The minimal cost of implementing low emissions is $h - la$
- The step costs of being in a bad world are $1 - b$
- $1 - b < h - la \Rightarrow$ reducing emissions is never a best choice!

Completing the specification

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- *State*, *Ctrl*, *next* and *reward* as discussed.

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- $crE = 4$ and $crN = 2$.
- $b = 0.5$, $lu = 0.1$, $la = 0.2$, $h = 0.3$.

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- *crE* $= 4 \Rightarrow$ it takes at least 5 steps to achieve states in which the sum of the cumulated emissions exceeds *crE* and, the probability of a transition to a bad world increases from *pS1* to *pS2*.
- *crN* $= 2 \Rightarrow$ it takes 3 steps to achieve states in which the probability that effective technologies for reducing GHG emissions become available increases from *pA1* to *pA2*.

Certain case: basic facts

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- Effective technologies become available (with certainty) after 4 steps.
- The state of the world turns bad (with certainty) after 6 steps at high emissions.
- For any given policy sequence there is exactly one possible state-control trajectory.

Certain case: policies

- *Const High* policies:

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  - trajectories, probabilities, rewards:

```
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )], 100%, 9.7
```

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     ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )], 100%, 9.7
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  - Expected sum of rewards = 9.7.

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     ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )], 100%, 9.7
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     ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )], 100%, 9.7
    ```
  - Expected sum of rewards = 9.7.
- *Const Low* policies:
  - trajectories, probabilities, rewards
    ```
    [((0,H,U,G),L), ((0,L,U,G),L), ((0,L,U,G),L), ((0,L,U,G),L), ((0,L,A,G),L),
     ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G), )], 100%, 10.5
    ```

Certain case: policies

- *Const High* policies:
  - trajectories, probabilities, rewards:

```
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )], 100%, 9.7
```

  - Expected sum of rewards = 9.7.
- *Const Low* policies:
  - trajectories, probabilities, rewards

```
[((0,H,U,G),L), ((0,L,U,G),L), ((0,L,U,G),L), ((0,L,U,G),L), ((0,L,A,G),L),
 ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G), )], 100%, 10.5
```

  - Expected sum of rewards = 10.5

Certain case: policies

- *Const High* policies:
  - trajectories, probabilities, rewards:
    `[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),`
    `((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )], 100%, 9.7`
  - Expected sum of rewards = 9.7.
- *Const Low* policies:
  - trajectories, probabilities, rewards
    `[((0,H,U,G),L), ((0,L,U,G),L), ((0,L,U,G),L), ((0,L,U,G),L), ((0,L,A,G),L),`
    `((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G), )], 100%, 10.5`
  - Expected sum of rewards = 10.5
- Optimal policies:

Certain case: policies

- *Const High* policies:
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    ```
    [((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
     ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )], 100%, 9.7
    ```

  - Expected sum of rewards = 9.7.
- *Const Low* policies:
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    ```
    [((0,H,U,G),L), ((0,L,U,G),L), ((0,L,U,G),L), ((0,L,U,G),L), ((0,L,A,G),L),
     ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G), )], 100%, 10.5
    ```

  - Expected sum of rewards = 10.5
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    ```
    [((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),L),
     ((4,L,A,G),L), ((4,L,A,G),L), ((4,L,A,G),L), ((4,L,A,G),H), ((5,H,A,G)  )], 100%, 11.3
    ```

Certain case: policies

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    `[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),`
    `((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )], 100%, 9.7`
  - Expected sum of rewards = 9.7.
- *Const Low* policies:
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    `[((0,H,U,G),L), ((0,L,U,G),L), ((0,L,U,G),L), ((0,L,U,G),L), ((0,L,A,G),L),`
    `((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G), )], 100%, 10.5`
  - Expected sum of rewards = 10.5
- Optimal policies:
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    `[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),L),`
    `((4,L,A,G),L), ((4,L,A,G),L), ((4,L,A,G),L), ((4,L,A,G),H), ((5,H,A,G)  )], 100%, 11.3`
  - Expected sum of rewards = 11.3

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    ```
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    [((0,H,U,G),L), ((0,L,U,G),L), ((0,L,U,G),L), ((0,L,U,G),L), ((0,L,A,G),L),
     ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G),L), ((0,L,A,G), )], 100%, 10.5
    ```
  - Expected sum of rewards = 10.5
- Optimal policies:
  - trajectories, probabilities, rewards

    ```
    [((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),L),
     ((4,L,A,G),L), ((4,L,A,G),L), ((4,L,A,G),L), ((4,L,A,G),H), ((5,H,A,G)  )], 100%, 11.3
    ```
  - Expected sum of rewards = 11.3
- Optimal policies dictate postponing emission reductions until effective technologies for reducing emissions become available!

Uncertainty on implementability: basic facts

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- $pS2 = pA1 = 0$, $pS1 = pA2 = 1$ but ...

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- $pS2 = pA1 = 0$, $pS1 = pA2 = 1$ but ...
- ... $pLL = pHH = 0.9$ and $pLH = pHL = 0.7$
- Effective technologies still become available after 4 steps and the state of the world turns bad after 6 steps at high emissions but ...

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- $pS2 = pA1 = 0$, $pS1 = pA2 = 1$ but ...
- ... $pLL = pHH = 0.9$ and $pLH = pHL = 0.7$
- Effective technologies still become available after 4 steps and the state of the world turns bad after 6 steps at high emissions but ...
- ... a policy (optimal or not) now entails $2^9 = 512$ possible trajectories.

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```
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )], 38.7%,  9.7
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((8,L,A,B)  )],  4.3%,  9.6
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((4,L,A,G),H), ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B), )],  3.3%, 10.1
...
```

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```
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )], 38.7%,  9.7
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((8,L,A,B)  )],  4.3%,  9.6
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((4,L,A,G),H), ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B), )],  3.3%, 10.1
...
```

  - Expected sum of rewards = 9.904.

Uncertainty on implementability: policies

- *Const High* policies:
  - trajectories, probabilities, rewards

```
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )], 38.7%,  9.7
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((8,L,A,B)  )],  4.3%,  9.6
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((4,L,A,G),H), ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B), )],  3.3%, 10.1
...
```

  - Expected sum of rewards $= 9.904$.
- Optimal policies:

Uncertainty on implementability: policies

- *Const High* policies:
  - trajectories, probabilities, rewards

```
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )], 38.7%,  9.7
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((8,L,A,B)  )],  4.3%,  9.6
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((4,L,A,G),H), ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B), )],  3.3%, 10.1
...
```

  - Expected sum of rewards = 9.904.

- Optimal policies:
  - trajectories, probabilities, rewards

```
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),L), ((2,L,U,G),L), ((2,L,A,G),L),
 ((2,L,A,G),L), ((2,L,A,G),H), ((3,H,A,G),H), ((4,H,A,G),H), ((5,H,A,G), )], 23.4%, 11.2
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),L), ((3,H,U,G),L), ((3,L,A,G),L),
 ((3,L,A,G),L), ((3,L,A,G),L), ((3,L,A,G),H), ((4,H,A,G),H), ((5,H,A,G), )],  7.8%, 11.3
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),L), ((2,L,U,G),L), ((2,L,A,G),L),
 ((2,L,A,G),L), ((2,L,A,G),H), ((2,L,A,G),H), ((3,H,A,G),H), ((4,H,A,G), )],  7.8%, 11.1
...
```

Uncertainty on implementability: policies

- *Const High* policies:
  - trajectories, probabilities, rewards

```
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )], 38.7%,  9.7
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((8,L,A,B)  )],  4.3%,  9.6
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((4,L,A,G),H), ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B), )],  3.3%, 10.1
...
```

  - Expected sum of rewards $= 9.904$.

- Optimal policies:
  - trajectories, probabilities, rewards

```
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),L), ((2,L,U,G),L), ((2,L,A,G),L),
 ((2,L,A,G),L), ((2,L,A,G),H), ((3,H,A,G),H), ((4,H,A,G),H), ((5,H,A,G), )], 23.4%, 11.2
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),L), ((3,H,U,G),L), ((3,L,A,G),L),
 ((3,L,A,G),L), ((3,L,A,G),L), ((3,L,A,G),H), ((4,H,A,G),H), ((5,H,A,G), )],  7.8%, 11.3
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),L), ((2,L,U,G),L), ((2,L,A,G),L),
 ((2,L,A,G),L), ((2,L,A,G),H), ((2,L,A,G),H), ((3,H,A,G),H), ((4,H,A,G), )],  7.8%, 11.1
...
```

  - Expected sum of rewards $= 11.085$.

Uncertainty on implementability: policies

- *Const High* policies:
  - trajectories, probabilities, rewards

```
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )], 38.7%,  9.7
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((8,L,A,B)  )],  4.3%,  9.6
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),H), ((4,H,A,G),H),
 ((4,L,A,G),H), ((5,H,A,G),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B), )],  3.3%, 10.1
...
```

  - Expected sum of rewards = 9.904.
- Optimal policies:
  - trajectories, probabilities, rewards

```
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),L), ((2,L,U,G),L), ((2,L,A,G),L),
 ((2,L,A,G),L), ((2,L,A,G),H), ((3,H,A,G),H), ((4,H,A,G),H), ((5,H,A,G), )], 23.4%, 11.2
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),L), ((3,H,U,G),L), ((3,L,A,G),L),
 ((3,L,A,G),L), ((3,L,A,G),L), ((3,L,A,G),H), ((4,H,A,G),H), ((5,H,A,G), )],  7.8%, 11.3
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),L), ((2,L,U,G),L), ((2,L,A,G),L),
 ((2,L,A,G),L), ((2,L,A,G),H), ((2,L,A,G),H), ((3,H,A,G),H), ((4,H,A,G), )],  7.8%, 11.1
...
```

  - Expected sum of rewards = 11.085.
- Uncertainty about the implementability of decisions dictates earlier emission reductions!

More uncertainties

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- What happens to optimal policies if we account for more uncertainties in the decision problem?

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- We want to estimate the impacts of:

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    - There is a small probability that technologies become available before 4 steps and a small probability that technologies do not become available available even after 4 steps!

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  - Uncertainty on the consequences of exceeding the critical cumulated emission threshold crE:

More uncertainties

- What happens to optimal policies if we account for more uncertainties in the decision problem?
- We want to estimate the impacts of:
  - Uncertainty on the availability of efficient technologies:
    - There is a small probability that technologies become available before 4 steps and a small probability that technologies do not become available available even after 4 steps!
  - Uncertainty on the consequences of exceeding the critical cumulated emission threshold crE:
    - There is a small probability that the world turns bad before 6 high emission steps and a small probability that the world doesn't turns bad even after crE has been exceeded!

Uncertainty on the availability of technologies

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- *pLL*, *pHH*, *pLH*, *pHL*, *pS1* and *pS2* as before but ...

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- ... $pA1 = 0.1$ and $pA2 = 0.9$ instead of 0 and 1.

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- $2^n * (n+1) = 5120$ possible trajectories for a policy sequence for $n = 9$ steps!

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- ... $pA1 = 0.1$ and $pA2 = 0.9$ instead of 0 and 1.
- $2^n * (n + 1) = 5120$ possible trajectories for a policy sequence for $n = 9$ steps!
- Optimal policies entail the same most likely trajectories. The expected sum of rewards is almost the same!

Uncertainty on the consequences of exceeding *crE*

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- *pLL*, *pHH*, *pLH*, *pHL*, *pA1* and *pA2* as before but ...

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Uncertainty on the consequences of exceeding *crE*

- *pLL*, *pHH*, *pLH*, *pHL*, *pA1* and *pA2* as before but ...
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- 51200 possible trajectories for a 9-steps policy sequence!

Uncertainty on the consequences of exceeding *crE*

- *pLL*, *pHH*, *pLH*, *pHL*, *pA1* and *pA2* as before but ...
- ... $pS1 = 0.9$ and $pS2 = 0.1$ instead of 1 and 0.
- 51200 possible trajectories for a 9-steps policy sequence!
- For *Const High* policies the most likely trajectory is unchanged but ...

Uncertainty on the consequences of exceeding *crE*

- Optimal policies look now quite different:

Uncertainty on the consequences of exceeding *crE*

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  - trajectories, probabilities, rewards

```
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),L), ((3,L,A,G),L),
 ((3,L,A,G),L), ((3,L,A,G),L), ((3,L,A,G),H), ((4,H,A,G),H), ((5,H,A,G),)],  5.9%, 11.3

[((0,H,U,G),H), ((1,H,U,B),H), ((2,H,U,B),H), ((3,H,U,B),H), ((4,H,A,B),H),
 ((5,H,A,B),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )],  2.5%,  7.2

[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,B),H), ((3,H,U,B),H), ((4,H,A,B),H),
 ((5,H,A,B),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )],  2.3%,  7.7
...
```

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```
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),L), ((3,L,A,G),L),
 ((3,L,A,G),L), ((3,L,A,G),L), ((3,L,A,G),H), ((4,H,A,G),H), ((5,H,A,G),)],  5.9%, 11.3

[((0,H,U,G),H), ((1,H,U,B),H), ((2,H,U,B),H), ((3,H,U,B),H), ((4,H,A,B),H),
 ((5,H,A,B),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )],  2.5%,  7.2

[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,B),H), ((3,H,U,B),H), ((4,H,A,B),H),
 ((5,H,A,B),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )],  2.3%,  7.7
...
```

  - Expected sum of rewards = 9.543

Uncertainty on the consequences of exceeding *crE*

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  - trajectories, probabilities, rewards

```
[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,G),H), ((3,H,U,G),L), ((3,L,A,G),L),
 ((3,L,A,G),L), ((3,L,A,G),L), ((3,L,A,G),H), ((4,H,A,G),H), ((5,H,A,G),)],  5.9%, 11.3

[((0,H,U,G),H), ((1,H,U,B),H), ((2,H,U,B),H), ((3,H,U,B),H), ((4,H,A,B),H),
 ((5,H,A,B),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )],  2.5%,  7.2

[((0,H,U,G),H), ((1,H,U,G),H), ((2,H,U,B),H), ((3,H,U,B),H), ((4,H,A,B),H),
 ((5,H,A,B),H), ((6,H,A,B),H), ((7,H,A,B),H), ((8,H,A,B),H), ((9,H,A,B), )],  2.3%,  7.7
...
```

  - Expected sum of rewards = 9.543

- Uncertainty about the consequences of exceeding *crE* make earlier emission reductions sub-optimal!

Wrap-up

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- Perhaps not surprisingly, more uncertainty about the implementability of decisions dictates earlier emission reductions. But . . .

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- Is this what you would have expected? Why?

Wrap-up

- Perhaps not surprisingly, more uncertainty about the implementability of decisions dictates earlier emission reductions. But . . .
- . . . more uncertainty about the implications of exceeding critical thresholds make earlier reductions sub-optimal!
- Is this what you would have expected? Why?
- The results are rigorous: optimality of "optimal" policies is machine-checked!

Thanks for your attention!

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- These slides: `https://gitlab.pik-potsdam.de/botta/IdrisLibs/tree/master/lectures/2018-12-03.PIK.Global_change_management_week`